

# On the existence of approximate equilibria and sharing rule solutions in discontinuous games

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This paper studies the existence of equilibrium solution concepts in a large class of economic models with discontinuous payoff functions. The issue is well understood for Nash equilibria, thanks to Reny's better-reply security condition (Reny 1999) and its recent improvements (Barelli and Meneghel 2013, McLennan et al. 2011, Reny 2009, 2011). We propose new approaches, related to Reny's work, and obtain tight conditions for the existence of approximate equilibria and of sharing rule solutions in pure and mixed strategies (Simon and Zame 1990). As byproducts, we prove that many auction games with correlated types admit an approximate equilibrium, and that many competition models have a sharing rule solution.

**KEYWORDS.** Discontinuous games, better-reply security, sharing rules, approximate equilibrium, Reny equilibrium, strategic approximation, auctions, timing games.

**JEL CLASSIFICATION.** C02, C62, C72.

## 1. INTRODUCTION

Many economic interactions are modeled as games with discontinuous payoff functions. For example, in timing games, price and spatial competitions, auctions, bargaining, preemption games, or wars of attrition, discontinuities occur when firms choose the same price, location, bid, or acting time. The objective of this paper is to extend and link conditions under which Nash, approximate, and sharing rule equilibria exist in such games.

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Nash equilibrium is a strategy profile where each agent is reacting optimally to other players' plans. Mathematically, it is a fixed point of the best-response correspondence. When payoff functions are continuous and quasiconcave, an application of Kakutani's fixed point theorem leads to the Nash–Glicksberg theorem (Glicksberg 1952, Nash 1950, 1951). In discontinuous games, the Kakutani approach cannot be directly applied because a player may have no optimal reply or because his best choice jumps as a function of the choices of the other players.<sup>1</sup>

A natural issue then is to identify regularity conditions on payoffs, which, combined with quasiconcavity of the payoff functions, guarantee the existence of a Nash equilibrium. The first existence conditions are given by the seminal papers of Dasgupta and Maskin (1986a, 1986b). A significant breakthrough<sup>2</sup> is achieved by Reny (1999) via the better-reply security approach.

Quoting Reny, “A game is better-reply secure if for every nonequilibrium strategy  $x^*$  and every payoff vector limit  $u^*$  resulting from strategies approaching  $x^*$ , some player  $i$  has a strategy yielding a payoff strictly above  $u_i^*$  even if the others deviate slightly from  $x^*$ .”

Reny's paper generated a large literature. For instance, Borelli and Meneghel (2013) and McLennan et al. (2011) proposed relaxations that cover non-quasiconcave preferences. Reny (2009, 2011) proposed new refinements for games in mixed strategies using a strategic approximation methodology. Recently, Borelli et al. (2014) applied Reny's better-reply security and strategic approximation techniques to prove the existence of the value in a Colonel Blotto game.

But what if a Nash equilibrium does not exist? Two related relaxations of Nash equilibrium have been studied in the literature: endogenous sharing rules and approximate equilibria.

In many discontinuous games, the *exogenously* given tie-breaking rule leads to games without pure Nash equilibria (e.g., asymmetric Bertrand duopoly, Hotelling location game) or without mixed Nash equilibria (e.g., three-player preemption games (Laraki et al. 2005), auctions with correlated types or values (Fang and Morris 2006, Jackson 2009)). However, the existence of a Nash equilibrium is restored if the tie-breaking rule is chosen *endogenously* (Andreoni et al. 2007, Maskin and Riley 2000, Simon and Zame 1990). For example, in an asymmetric Bertrand duopoly, a pure Nash equilibrium exists if ties are broken in favor of the lower cost firm. In first-price auctions with complete information, a pure Nash equilibrium exists if ties are broken in favor of the bidder with the highest value. Under mild topological conditions, Simon and Zame (1990) proved that to every game, one could associate an auxiliary game that admits a Nash equilibrium in mixed strategies and where payoffs in the two games only differ at discontinuity points (see Section 2 for a formal definition). Jackson et al. (2002) remark that their “results concern only the existence of solutions [sharing rule] in mixed strategies” and that they “have little to say about the existence of solutions in pure strategies.”

<sup>1</sup>Another approach, not considered in this paper, is to use ordered fixed point theory (e.g., Tarski's theorem) to obtain the existence of Nash equilibria in supermodular games (Topkis 1979).

<sup>2</sup>Carmona (2009) gives an extension of Dasgupta and Maskin's results, which is unrelated to Reny's approach.

We prove the existence of a sharing rule solution in pure strategies in every quasiconcave and compact game (see [Theorem 3.4](#)).

An alternative solution for games without a Nash equilibrium is the notion of approximate equilibrium. It is a limit strategy profile  $x^*$  and a limit payoff vector  $u^*$  of  $\varepsilon$ -Nash equilibria  $x^\varepsilon$  with associated payoff vector  $u(x^\varepsilon)$ , as  $\varepsilon$  goes to 0.

There are many games without a Nash equilibrium but with a reasonable approximate equilibrium. In first-price auctions with complete information, for example, a natural approximate equilibrium arises if the player with the highest value proposes a bid slightly above the second highest value, and if the other players bid exactly their value. Another example is Bertrand duopoly with asymmetric costs, where in every approximate equilibria in pure strategies the most efficient firm proposes a price slightly below the marginal cost of the opponent.

There are few results in the literature establishing the existence of approximate equilibria, one of which is due to [Reny \(1995\)](#) and [Prokopovych \(2011\)](#).<sup>3</sup> While theoretically interesting, it requires assumptions that are not always satisfied in applications (as will be seen).

In this paper, we define a game  $G$  to be *approximately better-reply secure* if for every nonapproximate equilibrium strategy profile  $x^*$  and every payoff vector limit  $u^*$  resulting from strategies approaching  $x^*$ , some player  $i$  has a strategy yielding a payoff strictly above  $u_i^*$ , even if the others deviate slightly from  $x^*$ .

We prove that every approximately better-reply secure quasiconcave compact game admits an approximate equilibrium. An example is given by the class of diagonal games, which encompasses many models of competition (in price, time, location, or quantity): each player,  $i = 1, \dots, N$ , chooses a real number  $x_i$  in  $[0, 1]$ . The payoff of player  $i$  is  $f_i(x_i, \phi(x_{-i}))$  if  $x_i < \phi(x_{-i})$ ,  $g_i(x_i, \phi(x_{-i}))$  if  $x_i > \phi(x_{-i})$ , and  $h_i(x_i, x_{-i})$  if  $x_i = \phi(x_{-i})$ , where  $f_i : [0, 1]^2 \rightarrow \mathbb{R}$ ,  $g_i : [0, 1]^2 \rightarrow \mathbb{R}$ , and  $\phi : [0, 1]^{N-1} \rightarrow \mathbb{R}$  are continuous functions. For example, in first-price auctions,  $f_i = 0$ ,  $g_i(x_i, x_{-i}) = v_i - x_i$ , and  $\phi(x_i, x_{-i}) = \max_{j \neq i} x_j$ . In second-price auctions,  $g_i(x_i, x_{-i}) = v_i - \max_{j \neq i} x_j$  (where  $v_i$  is the value of the object for player  $i$ ).

The paper is organized as follows. In [Section 2](#), we recall the main results for the existence of a solution in discontinuous games: Reny's better-reply security ([Reny 1999](#)) for the existence of Nash equilibria, the sharing rule solution of [Simon and Zame \(1990\)](#), and Reny and Prokopovych's conditions ([Reny 1995](#), [Prokopovych 2011](#)) for the existence of an approximate equilibrium. [Section 3](#) is dedicated to quasiconcave compact games in *pure strategies*. We introduce the new concept of Reny solution, which weakens the Nash equilibrium concept, and we prove its existence for every discontinuous game. This solution is used to construct pure sharing rule solutions and approximate equilibria. The results are illustrated in the class of diagonal games. [Section 4](#) is dedicated to compact metric games in *mixed strategies*. We prove that the intersection of the sets of Reny solutions and sharing rule solutions is nonempty and contains the set of approximate equilibria. In addition, we prove that in every approximately better-reply secure

<sup>3</sup>Historically, Reny proved this result in an unpublished working paper ([Reny 1995](#)). Independently, [Prokopovych \(2011\)](#) proved the same result with a different technique.

game, approximate equilibria may be obtained as limits of Nash equilibria of an *endogenously* chosen sequence of discretizations of the game. This is a natural extension of a similar result established by Reny (2009, 2011) for Nash equilibria. As applications, we prove the existence of a mixed approximate equilibrium in multi-player auctions with correlated types and in two-player auctions with interdependent types and values.

## 2. THREE STANDARD APPROACHES TO DISCONTINUOUS GAMES

A game in strategic form  $G = ((X_i)_{i \in N}, (u_i)_{i \in N})$  is given by a finite set  $N$  of players, and for each player  $i \in N$ , a nonempty set  $X_i$  of pure strategies and a payoff function  $u_i : X = \prod_{i \in N} X_i \rightarrow \mathbb{R}$ . This paper assumes  $G$  to be *compact*, that is, for every  $i$ ,  $X_i$  is a compact subset of a topological vector space, and  $u_i$  is bounded. We let  $V_i(x_{-i}) := \sup_{d_i \in X_i} u_i(d_i, x_{-i})$  denote the greatest payoff that player  $i$  can get against  $x_{-i} = (x_j)_{j \neq i} \in X_{-i} := \prod_{j \neq i} X_j$ .

**DEFINITION 2.1.** A pair  $(x, v) \in X \times \mathbb{R}^N$  is a *Nash equilibrium* of  $G$  (and  $x$  is a *Nash equilibrium profile*) if  $v = u(x)$  and for every player  $i \in N$ ,  $V_i(x_{-i}) = v_i$ .

The game  $G$  is *quasiconcave* if for every player  $i \in N$ ,  $X_i$  is convex and for every  $x_{-i} \in X_{-i}$ , the mapping  $u_i(\cdot, x_{-i})$  is quasiconcave. The game is *continuous* if for every  $i \in N$ ,  $u_i$  is a continuous function.<sup>4</sup>

**THEOREM 2.2 (Glicksberg 1952).** *Every continuous, quasiconcave, and compact game admits a Nash equilibrium.*

The rest of the section presents three extensions of this result when payoffs are discontinuous. Our paper combines them into one general idea.

### 2.1 Better-reply secure game

In many discontinuous games, a Nash equilibrium exists (symmetric Bertrand competition, auctions with private values, and wars of attrition, among many). Reny's theorem provides an explanation for this (Reny 1999). Formally, we let  $\Gamma = \{(x, u(x)) : x \in X\}$  denote the graph of  $G$  and let  $\bar{\Gamma}$  be the closure of  $\Gamma$ . Since  $X$  is compact and  $u$  is bounded,  $\bar{\Gamma}$  is a compact subset of  $X \times \mathbb{R}^N$ . Define the "secure payoff level" of player  $i$  at  $(d_i, x_{-i}) \in X$  as

$$\underline{u}_i(d_i, x_{-i}) := \liminf_{x'_{-i} \rightarrow x_{-i}} u_i(d_i, x'_{-i}) := \sup_{V \in \mathcal{V}(x_{-i})} \inf_{x'_{-i} \in V} u_i(d_i, x'_{-i}),$$

where  $\mathcal{V}(x_{-i})$  denotes the set of neighborhoods of  $x_{-i}$ . This is the payoff that  $d_i$  can almost guarantee to player  $i$  if his opponents play any profile close enough to  $x_{-i}$ . We let  $\underline{V}_i(x_{-i}) := \sup_{d_i \in X_i} \underline{u}_i(d_i, x_{-i})$  denote the largest payoff that player  $i$  can secure against  $x_{-i}$ .

<sup>4</sup>Set  $X$  is endowed with the product topology.

DEFINITION 2.3. A game  $G$  is *better-reply secure* if whenever  $(x, v) \in \bar{\Gamma}$  and  $x$  is not a Nash equilibrium profile, some player  $i \in N$  can secure<sup>5</sup> a payoff strictly above  $v_i$ , i.e.,  $\underline{V}_i(x_{-i}) > v_i$ .

THEOREM 2.4 (Reny 1999). *Every better-reply secure, quasiconcave, and compact game admits a Nash equilibrium in pure strategies.*

Since every continuous game is obviously better-reply secure, this extends Glicksberg's theorem. In his paper, Reny gives two practical conditions that, together, imply better-reply security (see Theorem 2.6 below).

DEFINITION 2.5. (i) Game  $G$  is payoff secure if  $V_i(x_{-i}) = \underline{V}_i(x_{-i})$ .

(ii) Game  $G$  is reciprocally upper semicontinuous if, whenever  $(x, v) \in \bar{\Gamma}$  and  $u_i(x) \leq v_i$  for every  $i$ , then  $u(x) = v$ .

Equivalently,  $G$  is payoff secure if for every  $x \in X$  and for every  $\varepsilon > 0$ , every player  $i \in N$  can secure a payoff above  $u_i(x) - \varepsilon$ .

THEOREM 2.6. *Every payoff secure and reciprocally upper semicontinuous game is better-reply secure.*

## 2.2 Approximate equilibrium

In first-price auctions with complete information, bidding slightly above the second highest evaluation for the bidder with the highest evaluation yields an approximate equilibrium. One of the main goals of this paper is to develop theoretical tools to extend this existence result to a large class of auctions.

DEFINITION 2.7. A pair  $(x, v) \in \bar{\Gamma}$  is an *approximate equilibrium* (and  $x$  is an approximate equilibrium profile) if there exists a sequence  $(x^n)_{n \in \mathbb{N}}$  of  $X$  and a sequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  of positive real numbers, converging to 0, such that the following statements hold:

(i) For every  $n \in \mathbb{N}$ ,  $x^n$  is an  $\varepsilon_n$ -equilibrium:  $\forall i \in N, \forall d_i \in X_i, u_i(d_i, x_{-i}^n) \leq u_i(x^n) + \varepsilon_n$ .

(ii) The sequence  $(x^n, u(x^n))$  converges to  $(x, v)$ .

In zero-sum games, the existence of an approximate equilibrium is equivalent to the existence of the value. Every Nash equilibrium is obviously an approximate equilibrium (take a constant sequence in the definition above). Let us state one of the very few existing results in the literature.<sup>6</sup>

<sup>5</sup>The following definitions are standard: Player  $i$  can secure a payoff above  $\alpha \in \mathbb{R}$  if there exists  $d_i \in X_i$  and a neighborhood  $V_{-i}$  of  $x_{-i}$  such that for every  $x'_{-i} \in V_{-i}$ ,  $u_i(d_i, x'_{-i}) \geq \alpha$ . Player  $i$  can secure a payoff strictly above  $\alpha \in \mathbb{R}$  if he can secure a payoff above  $\alpha + \varepsilon$  for some  $\varepsilon > 0$ . We give equivalent formulations using  $V_i$  and  $\underline{V}_i$ .

<sup>6</sup>Ziad (1997) proposes another existence theorem of approximate equilibria, unrelated to our work. See also Carmona (2010, 2011) and Reny (1995).

**DEFINITION 2.8.** A game  $G$  has the marginal continuity property at  $x \in X$  if for every  $i \in N$ ,  $V_i(x_{-i})$  is a continuous function at  $x_{-i}$ . If this holds for every  $x$ , the game has the marginal continuity property.

**THEOREM 2.9** (Reny 1995 and Prokopovych 2011). *Every payoff secure, quasiconcave compact game that has the marginal continuity property admits an approximate equilibrium.*

This theorem applies to first-price auctions and asymmetric Bertrand duopoly. However, the following location game (Simon and Zame 1990) is not payoff secure, but admits an approximate equilibrium.

**EXAMPLE 2.10** (California location game). This example was introduced by Simon and Zame (1990). The length interval  $[0, 4]$  represents an interstate highway. The strategy set of player 1 (a psychologist from California) is  $X = [0, 3]$  (representing the Californian highway stretch). The strategy set of player 2 (a psychologist from Oregon) is  $Y = [3, 4]$  (the Oregon part of the highway). The payoff function of player 1 is  $u_1(x, y) = (x + y)/2$  if  $x < y$  and  $u_1(3, 3) = 2$ . The payoff function of player 2 is  $u_2(x, y) = 4 - u_1(x, y)$ . The strategy profile  $x_n = (3 - 1/n, 3)$ , corresponding to the vector payoff  $v_n = (3 - 1/(2n), 1 + 1/(2n))$ , is a  $1/(2n)$  equilibrium. Consequently,  $(x = (3, 3), v = (3, 1))$  is an approximate equilibrium. However, the game is not payoff secure for player 2 at  $x = (3, 3)$ .  $\diamond$

### 2.3 Sharing rule solutions

Simon and Zame show that even if a game does not have a Nash equilibrium, it is always possible to slightly change the payoffs at discontinuity points so that the new game has a mixed Nash equilibrium.

**EXAMPLE 2.11** (California location game, continued). In the California location game, define a new payoff function  $q$  as  $q(x) = u(x)$  for every  $x \neq (3, 3)$  and  $q(3, 3) = (3, 1)$ . The pure strategy profile  $(3, 3)$  with payoff  $(3, 1)$  is a Nash equilibrium of the game defined by  $q$ . The new sharing rule at  $x = (3, 3)$  has a simple interpretation: it corresponds to giving each psychologist his/her natural market share. Moreover, this is exactly the prediction of the approximate equilibrium in Example 2.10. We will prove that this property is very general: every approximate equilibrium is a sharing rule equilibrium (see Theorem 3.10).  $\diamond$

To prove the existence of a solution, Simon and Zame do not require the game to be quasiconcave. However, they allow the use of mixed strategies. Formally,  $G$  is *metric* if strategy sets are Hausdorff and metrizable and payoff functions are measurable. Denote by  $M_i = \Delta(X_i)$  the set of Borel probability measures on  $X_i$  (usually called the set of mixed strategies of player  $i$ ). This is a compact Hausdorff metrizable set under the weak\* topology. Let  $M = \prod_i M_i$ .

DEFINITION 2.12. A *mixed Nash equilibrium* of  $G$  is a pure Nash equilibrium of its mixed extension  $G' = ((M_i)_{i \in N}, (u_i)_{i \in N})$ , where payoff functions are extended multilinearly to  $M$ .

DEFINITION 2.13. A pair  $(m, q)$  is a *mixed sharing rule solution* of  $G$  if  $m \in M$  is a mixed Nash equilibrium of the auxiliary game  $\tilde{G} = ((X_i)_{i \in N}, (q_i)_{i \in N})$ , where the auxiliary measurable payoff functions  $q = (q_i)_{i \in N}$  must satisfy the condition

$$(SR) \quad \forall x \in X, \quad q(x) \in \text{co} \bar{\Gamma}_x,$$

where  $\bar{\Gamma}_x = \{v \in \mathbb{R}^N : (x, v) \in \bar{\Gamma}\}$  is the  $x$  section of  $\bar{\Gamma}$  and  $\text{co}$  stands for the convex hull.

Condition (SR) has two implications: if  $u$  is continuous at  $x$ ,  $q(x) = u(x)$ ; if  $\sum_{i \in N} u_i(x)$  is continuous, then  $\sum_{i \in N} q_i(x) = \sum_{i \in N} u_i(x)$  (justifying the terminology “sharing rule”).

THEOREM 2.14 (Simon and Zame 1990). *Every compact metric game admits a mixed sharing rule solution.*

Jackson et al. (2002) extend Simon and Zame’s theorem to games with incomplete information. In their paper, they interpret a tie-breaking rule as a proxy for the outcome of an unmodeled second stage game. As an example, they recall the analysis of first-price auctions with incomplete information for a single indivisible object. Maskin and Riley (2000) add to the sealed-bid stage a second stage where bidders with the greatest bid in the first stage play a Vickrey auction. In the private value setting, their dominant strategy is to bid their true values. Consequently, the second stage induces a tie-breaking rule where the bidder with the highest value wins the object. More generally, a tie-breaking rule may be implemented by asking players to send a cheap message (their private values in auctions) in addition to their strategies (bids). The messages will be used only to break ties (as in the second stage of Maskin and Riley’s mechanism).

When the game is continuous, the new and the original games coincide, and so we recover the Nash–Glicksberg theorem in mixed strategies.

THEOREM 2.15 (Nash–Glicksberg theorem in mixed strategies). *Every continuous, metric compact game admits a mixed Nash equilibrium.*

In the next section, we prove the existence of a pure sharing rule solution, which we define now.

DEFINITION 2.16. A pair  $(x, q)$  is a *pure sharing rule solution* if  $x \in X$  is a pure Nash equilibrium of the auxiliary game  $\tilde{G} = ((X_i)_{i \in N}, (q_i)_{i \in N})$ , where the auxiliary payoff functions  $q = (q_i)_{i \in N}$  must satisfy the condition<sup>7</sup>

$$(SR \text{ strong}) \quad \forall y \in X, \quad q(y) \in \bar{\Gamma}_y.$$

<sup>7</sup>If  $(m, q)$  is a mixed sharing rule solution and  $m \in X$  is also a pure strategy, then  $(m, q)$  may not be a pure sharing rule solution (because  $q$  may not satisfy (SR strong)), but the converse implication is always true.

If  $G$  is metric, condition (SR strong) requires that for every strategy profile  $y$ , there exists a sequence  $(y_n)$  converging to  $y$  such that  $q(y) = \lim_n u(y_n)$ . On one hand, our condition is stronger than the original condition (SR) because one always has  $\bar{\Gamma}_y \subset \text{co } \bar{\Gamma}_y$ . On the other hand, to prove the existence of a pure sharing rule solution, we need payoff functions to be quasiconcave.

To allow comparison between sharing rule solution and approximate equilibrium, we introduce the following terminology.

**DEFINITION 2.17.** A pair  $(m, v) \in M \times \mathbb{R}^N$  (resp.  $(x, v) \in X \times \mathbb{R}^N$ ) is called a mixed (resp. pure) sharing rule equilibrium if  $(m, q)$  is a mixed (resp. pure) sharing rule solution for some  $q$  and  $q(m) = v$ .

The proof of the existence of a pure sharing rule equilibrium is a direct consequence of the existence of a Reny solution, which is defined in the next subsection.

### 3. EXISTENCE OF SOLUTIONS FOR GAMES IN PURE STRATEGIES

As discussed above, sharing rule and approximate equilibrium concepts are alternative solutions for games without a Nash equilibrium. Both are defined on  $\bar{\Gamma}$  (the closure of the graph of the game). To prove their existence, we use a new concept—Reny solution—defined on  $\bar{\Gamma}$ .

#### 3.1 Existence of a reny solution

In the following definition, recall that  $\underline{V}_i(x_{-i}) := \sup_{d_i \in X_i} u_i(d_i, x_{-i})$ .

**DEFINITION 3.1.** A pair  $(x, v) \in \bar{\Gamma}$  is a *Reny solution* if for every  $i \in N$ ,  $\underline{V}_i(x_{-i}) \leq v_i$ . The strategy profile  $x \in X$  is a Reny solution profile if  $(x, v) \in \bar{\Gamma}$  is a *Reny solution* for some  $v$ .

**EXAMPLE 3.2** (Two-player first-price auctions). Both players  $i = 1, 2$  choose a bid  $x_i \in [0, 1]$  and receive a payoff:

$$u_i(x_i, x_j) = \begin{cases} w_i - x_i & \text{if } x_i > x_j \\ \frac{w_i - x_i}{2} & \text{if } x_i = x_j \\ 0 & \text{if } x_i < x_j. \end{cases} \quad \diamond$$

If  $w_1 \in ]0, 1[$  (the value of player 1) is higher than  $w_2 \in ]0, 1[$  (the value of player 2), then the above game is quasiconcave, and every  $(x_1, x_2, v_1, v_2) = (y, y, w_1 - y, 0)$  is a Reny solution whenever  $y \in [w_2, w_1]$ . To see this, note first that the game is payoff secure; thus a Reny solution  $(x, v) = (x_1, x_2, v_1, v_2) \in \bar{\Gamma}$  satisfies

$$\sup_{d_i \in [0, 1]} u_i(d_i, x_{-i}) \leq v_i, \quad i = 1, 2. \quad (1)$$

Since this game has no Nash equilibrium,  $x_1$  is equal to  $x_2$  (otherwise  $u_i$  would be continuous at  $x = (x_1, x_2)$ , and (1) would imply that  $x$  is a Nash equilibrium). Moreover, each player can get a payoff of at least 0 by playing 0. Consequently,  $v_1$  and  $v_2$  are



nonnegative. From  $(x, v) \in \bar{\Gamma}$ ,  $(v_1, v_2) = \lim_{n \rightarrow +\infty} (u_1(x^n), u_2(x^n))$  for some sequence of profiles  $x^n = (x_1^n, x_2^n)$  converging to  $(x_1, x_1)$ . There are three cases (up to a subsequence), depending on whether the sequence converges to  $x$  from above, along the diagonal, or from below. In the two first cases, we get  $v = (0, w_2 - x_1)$  or  $v = ((w_1 - x_1)/2, (w_2 - x_1)/2)$ ; thus  $x_2 = x_1 \leq w_2 < 1$  (since  $v_2$  is nonnegative). Then, playing slightly above  $x_1$  gives a payoff strictly above  $v_1$  for player 1, which contradicts (1). In the last case,  $v = (w_1 - x_1, 0)$ ; thus  $x_1 \leq w_1$ . Then (1) implies that  $x_1 \geq w_2$  (otherwise player 2 could do better than 0 by playing slightly above  $x_1$ ). Conversely, it is easy to check that  $(y, y, w_1 - y, 0)$  is a Reny solution when  $y \in [w_2, w_1]$ .

In this example, the set of Reny solutions coincides with the set of approximate equilibria (playing  $y \in [w_2, w_1]$  for player 2 and slightly above for player 1 is an  $\varepsilon$ -equilibrium). Note that there are several Reny solutions and thus several approximate equilibria, but multiplicity also happens for Nash equilibria: in this example with a second-price auction mechanism, playing  $y$  for player 2 and  $w_1$  for player 1 is a Nash equilibrium for all  $y \in [0, w_1]$ .

**THEOREM 3.3.** *For every quasiconcave and compact game  $G$ , the set of Reny solutions is nonempty and compact, and it contains the set of Nash equilibria. Moreover,  $G$  is better-reply secure if and only if Nash equilibrium profiles and Reny solution profiles coincide.*

Observe that a Nash equilibrium  $(x, u(x))$  is a Reny solution because  $\underline{V}_i(x_{-i}) \leq V_i(x_{-i})$ , and by Nash conditions,  $V_i(x_{-i}) \leq u_i(x)$ . Moreover, if the game is continuous, then Reny solutions and Nash equilibria coincide because  $\underline{V}_i(x_{-i}) = V_i(x_{-i})$  and  $u(x) = v$ .

**PROOF OF THEOREM 3.3.** The existence of a Reny solution is a straightforward consequence of Reny's theorem (Reny 1999). Indeed, assume, by contradiction, that there is no Reny solution. This implies that the game is better-reply secure. Consequently, by Reny's theorem, there exists a Nash equilibrium, which is a Reny solution: a contradiction. Compactness of the set of Reny solutions is due to the lower semicontinuity of  $\underline{V}_i$ , and the last assertion of Theorem 3.3 is a consequence of the definition of better-reply security. Actually, if the game is better-reply secure, a Reny solution profile must be a Nash equilibrium profile because otherwise, better-reply security contradicts Reny solution condition. Conversely, if Nash equilibrium profiles and Reny solution profiles coincide, then for every  $x \in X$  that is not a Nash equilibrium profile,  $x$  is not a Reny solution profile; thus better-reply security condition is satisfied.  $\square$

As an illustration, we can revisit the result of Reny in Theorem 2.6 and prove that whenever a game is payoff secure and reciprocally upper semicontinuous, it is better-reply secure. Actually, assume  $(x, v)$  to be a Reny solution. Thus, for every  $i \in N$ ,  $\underline{V}_i(x_{-i}) \leq v_i$ . Since the game is payoff secure,  $V_i(x_{-i}) = \underline{V}_i(x_{-i}) \leq v_i$ . Since  $u_i(x) \leq \underline{V}_i(x_{-i})$ , one has  $u_i(x) \leq v_i$  for every  $i \in N$ . By reciprocal upper semicontinuity,  $v = u(x)$ , and so  $V_i(x_{-i}) \leq u_i(x)$  for every  $i \in N$ . Consequently,  $(x, v)$  is a Nash equilibrium.

Two major applications of the Reny solution are presented in the next subsections. In Section 3.2, the Reny solution allows the existence of a pure sharing rule equilibrium

to be proved in every quasiconcave compact game. In [Section 3.3](#), the Reny solution is used to prove the existence of approximate equilibria in a number of economic models.

### 3.2 Existence of a sharing rule equilibrium

The existence of a Reny solution allows the open problem in [Jackson et al. \(2002\)](#) to be solved.

**THEOREM 3.4.** *Every Reny solution is a pure sharing rule equilibrium. In particular, every quasiconcave and compact game  $G$  admits a pure sharing rule solution.*

**REMARK 3.5.** Observe that a pure sharing rule solution  $(m, q')$  of  $G'$  (the mixed extension of  $G$ ) is not a mixed sharing rule solution à la Simon and Zame of  $G$  because the new payoff profile  $q'$  is defined on  $M$  and not on  $X$ , and  $q'$  is not necessarily the multilinear extension of a pure strategy payoff profile. Thus, our result does not imply the Simon–Zame theorem.

**PROOF OF THEOREM 3.4.** To prove [Theorem 3.4](#), consider a Reny solution  $(x, v) \in \bar{\Gamma}$ . Then we can build a sharing rule solution as follows. For every  $i \in N$  and  $d_i \in X_i$ , denote by  $\underline{\mathcal{S}}(d_i, x_{-i})$  the space of sequences<sup>8</sup>  $(x_{-i}^n)_{n \in \mathbb{N}}$  of  $X_{-i}$  converging to  $x_{-i}$  such that  $\lim_{n \rightarrow +\infty} u_i(d_i, x_{-i}^n) = \underline{u}_i(d_i, x_{-i})$ . Then define  $q : X \rightarrow \mathbb{R}^N$  by

$$q(y) = \begin{cases} v & \text{if } y = x \\ \text{any limit point of } (u(d_i, x_{-i}^n))_{n \in \mathbb{N}} & \\ & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N, d_i \neq x_i, (x_{-i}^n)_{n \in \mathbb{N}} \in \underline{\mathcal{S}}(d_i, x_{-i}) \\ q(y) = u(y) & \text{otherwise.} \end{cases}$$

Now let us prove that  $(x, q)$  is a pure sharing rule solution. Since  $(x, v) \in \bar{\Gamma}$ , and by definition of  $q$ , condition (SR strong) of [Definition 2.16](#) is satisfied at  $x$  (this would be false if the definition of  $q$  in the second case above was  $q_j(d_i, x_{-i}) = \underline{u}_j(d_i, x_{-i})$ ,  $j = 1, \dots, n$ ). Obviously, it is satisfied at every  $y$  different from  $x$  for at least two components, and also at every  $(d_i, x_{-i})$  with  $d_i \neq x_i$ , from the definition of  $q(d_i, x_{-i})$  in this case.  $\square$

**REMARK 3.6.** Thus, Reny solution refines pure sharing rule equilibrium, and the refinement is strict as the following better-reply secure game shows. A player maximizes over  $[0, 1]$  the discontinuous payoff function  $u(x) = 0$  if  $x < 1$  and  $u(1) = 1$ . If  $q(y) = 0$  for every  $y$ , then  $(x, q)$  is a pure sharing rule solution for every  $x \in [0, 1]$ . Yet the only Reny solution is  $(x, v) = (1, 1)$ , and it coincides with the unique Nash equilibrium.

**REMARK 3.7.** An important question, raised by [De Castro \(2011\)](#) and [Carmona and Podczeck \(2014\)](#), is whether for every game  $G$  one can define a new payoff function  $q(y) \in \text{co}\bar{\Gamma}(y)$ , inducing a better-reply secure game. The mapping  $q$  defined in the proof

<sup>8</sup>If  $X$  is not first countable, one should consider the space of nets instead of sequences.

of [Theorem 3.4](#) does not answer the question that if  $(x, v) \in \bar{\Gamma}$  is the Reny solution considered in this proof, then  $q$  is equal to the initial payoff mapping  $u$  at every  $y \in X$  different from  $x$  for at least two components. Since  $u$  is arbitrarily discontinuous, there is no hope, in general, that  $q$  defines a better-reply secure game.

[De Castro \(2011\)](#) proposes a first answer: he introduces a regularity property—a weakening of better-reply security—as follows: a game is regular if for every  $(x, v) \in \bar{\Gamma}$ , if  $v_i \geq V_i(x_{-i})$  for all  $i \in N$ , then  $u_i(x) = V_i(x_{-i})$  for all  $i \in N$ . Then he proves that every discontinuous game has a measurable selection  $q(y) \in \text{co} \bar{\Gamma}(y)$  that induces a regular game. This result cannot be applied to prove [Theorem 3.4](#) because a regular game may have no Nash equilibrium.

Another approach is given by [Carmona and Podczeck \(2014\)](#) through the concept of virtual continuity. However, they consider the case of better-reply security *in mixed strategies*. Consequently, this is not directly related to [Theorem 3.4](#).

**REMARK 3.8.** Actually, the pure sharing rule solution  $(x, q)$  built in the proof of [Theorem 3.4](#) satisfies the additional property<sup>9</sup>  $q_i(d_i, x_{-i}) \geq \underline{u}_i(d_i, x_{-i})$  for every  $i \in N$  and every  $d_i \in X_i$ . This property says that  $q$  remains above the secure payoff level in the original game.

**APPLICATION 3.9** (Shared resource games). The payoff of each player  $i \in N$  can be written as  $u_i(x_i, x_{-i}) = F_i(x_i, S_i(x_i, x_{-i}))$ , where  $F_i : X_i \times \mathbb{R} \rightarrow \mathbb{R}$  and  $S_i : X \rightarrow \mathbb{R}$  (the shared resource of player  $i$ ). The total amount of the resource  $\sum_{i=1}^N S_i$  is a (possibly discontinuous) function of the strategy profile  $x \in X$ . This game  $G$  was introduced to model fiscal competition for mobile capital ([Rothstein 2007](#)).

A sharing rule of  $G$  is defined to be a family  $(\tilde{S}_i)_{i \in N}$  of functions from  $X$  to  $\mathbb{R}$  such that for every strategy profile  $x \in X$ , there is a sequence  $(x_n)$  converging to  $x$  such that for every player  $i$ ,  $\tilde{S}_i(x) = \lim_{n \rightarrow \infty} S_i(x_n)$ . [Theorem 3.4](#) implies the following extension of Rothstein's results. Assuming  $G$  to be quasiconcave and compact,  $F_i$  continuous, and  $S_i$  bounded for every player  $i$ , we get the existence of a new sharing rule  $(\tilde{S}_i)_{i \in N}$  whose associated game  $\tilde{u}_i(x_i, x_{-i}) = F_i(x_i, \tilde{S}_i(x_i, x_{-i}))$  admits a pure Nash equilibrium. Moreover, under the following assumptions, every Nash equilibrium of  $\tilde{G}$  is a Nash equilibrium<sup>10</sup> of  $G$ :

- A1. For all  $x_i \in X_i$ ,  $F_i(x_i, s_i)$  is nondecreasing in  $s_i$ .
- A2. For all  $x_{-i} \in X_{-i}$ ,  $\sup_{d_i \in X_i: (d_i, x_{-i}) \in \mathcal{C}_i} u_i(d_i, x_{-i}) = \sup_{d_i \in X_i} u_i(d_i, x_{-i})$ , where  $\mathcal{C}_i$  is the set of continuity points of  $S_i$ .
- A3. If  $x \notin \bigcap_{i \in N} \mathcal{C}_i$ ,  $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) > F_i(x_i, \limsup_{x' \rightarrow x} \sum_{i=1}^N S_i(x')/N)$  for every  $i \in N$ .

<sup>9</sup>Indeed, if  $d_i = x_i$ , then  $q_i(d_i, x_{-i}) = q_i(x) = v_i \geq \underline{u}_i(x)$  because  $(x, v)$  is a Reny solution. If  $d_i \neq x_i$ , then  $q_i(d_i, x_{-i}) = \underline{u}_i(d_i, x_{-i})$ .

<sup>10</sup>Indeed, for every Nash equilibrium  $x$  of  $\tilde{G}$ , one has  $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) \leq \sup_{d_i \in X_i} F_i(d_i, \tilde{S}_i(d_i, x_{-i})) \leq F_i(x_i, \tilde{S}_i(x))$  for every player  $i$ . The first inequality is a consequence of A2 and the definition of  $\tilde{S}_i$ . If  $x \in \mathcal{C}_i$  for every  $i$ , then  $\tilde{S}_i(x) = S_i(x)$ , and  $x$  is a Nash equilibrium of the initial game  $G$ . Otherwise, from A1 and A3, we get  $\limsup_{x' \rightarrow x} \sum_{i=1}^N S_i(x')/N < \tilde{S}_i(x)$  for every  $i \in N$ . Summing these inequalities contradicts the definition of  $\tilde{S}_i$ .

In his paper, Rothstein says “the work of [Simon and Zame \(1990\)](#) is directly applicable, but only to establish the existence of a Nash equilibrium in mixed strategies with an endogenous sharing rule.” By [Theorem 3.4](#), the game has a pure endogenous sharing rule equilibrium.

### 3.3 Existence of an approximate equilibrium

**THEOREM 3.10.** *Every approximate equilibrium is a Reny solution and, therefore, also a pure sharing rule equilibrium.*

**PROOF.** Let  $(x^n)_{n \in \mathbb{N}}$  be a sequence of  $\varepsilon_n$  equilibria such that  $(x^n, u(x^n))$  converges to  $(x, v)$ . By definition,  $u_i(d_i, x_{-i}^n) \leq u_i(x^n) + \varepsilon_n$  for every  $n \in \mathbb{N}$ , every player  $i \in N$ , and every deviation  $d_i \in X_i$ . Passing to the infimum limit when  $n$  tends to infinity, we obtain  $\underline{u}_i(d_i, x_{-i}) \leq v_i$ . Thus,  $(x, v)$  is a Reny solution, and also a pure sharing rule equilibrium by [Theorem 3.4](#).  $\square$

This leads to the following definition.

**DEFINITION 3.11.** A game  $G$  is *approximately better-reply secure* if whenever  $(x, v) \in \bar{\Gamma}$  and  $x$  is not an approximate equilibrium profile, some player  $i$  can secure a payoff strictly above  $v_i$ .

The existence of a Reny solution implies the following result.

**THEOREM 3.12.** *Every approximately better-reply secure quasiconcave and compact game admits an approximate equilibrium.*

The California location game is approximately better-reply secure. This theorem provides a local version of Reny–Prokopovych’s theorem (described in [Section 2.2](#)).

**COROLLARY 3.13.** *If  $(x, v)$  is a Reny solution and if, at  $x$ , the game is payoff secure and has the marginal continuity property, then  $(x, v)$  is an approximate equilibrium.*

**PROOF.** Actually, if  $(x, v)$  is a Reny solution, then

$$\sup_{d_i \in X_i} \underline{u}_i(d_i, x_{-i}) = \sup_{d_i \in X_i} u_i(d_i, x_{-i}) \leq v_i,$$

the equality being a consequence of payoff security at  $x$ . Since  $v = \lim_{x^n \rightarrow x} u(x^n)$  for some sequence  $x^n$ , the local continuity of  $\sup_{d_i \in X_i} u_i(d_i, x_{-i})$  with respect to  $x$  guarantees that  $(x, v)$  is an approximate equilibrium.  $\square$

This corollary implies Reny–Prokopovych’s theorem, and is useful in practice as the following application shows.

APPLICATION 3.14 (Diagonal games). For every  $i \in N$ , we let  $f_i, g_i$  be *continuous* mappings from  $[0, 1] \times [0, 1]$  to  $\mathbb{R}$ , and let  $h_i : [0, 1]^N \rightarrow \mathbb{R}$  be a bounded mapping. The payoff of player  $i$  is

$$u_i(x_i, x_{-i}) = \begin{cases} f_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) > x_i \\ g_i(x_i, \phi(x_{-i})) & \text{if } \phi(x_{-i}) < x_i \\ h_i(x_i, x_{-i}) & \text{if } \phi(x_{-i}) = x_i, \end{cases}$$

where  $\phi : [0, 1]^{N-1} \rightarrow [0, 1]$  is a continuous aggregation function, i.e., a function that aggregates the  $N - 1$  strategies of the opponents into one strategy in  $[0, 1]$ . This aggregation function must satisfy<sup>11</sup> for every  $(y, z) \in [0, 1]^{N-1} \times [0, 1]^{N-1}$  and  $t \in [0, 1]$  the following conditions:

Monotonicity. If  $y \leq z$ , then  $\phi(y) \leq \phi(z)$ ; if  $y \ll z$  then  $\phi(y) < \phi(z)$ .

Anonymity. For every permutation  $\sigma$  of  $\{1, \dots, N - 1\}$ , we have  $\phi(y_1, \dots, y_{N-1}) = \phi(y_{\sigma(1)}, \dots, y_{\sigma(N-1)})$ .

Unanimity. We have  $\phi(t, \dots, t) = t$ .

Representativity.<sup>12</sup> If  $\phi(y) > 0$  and  $y_i > 0$  for some  $i$ , then  $\phi(z_i, y_{-i}) > 0$  for every  $z_i > 0$ . Similarly, if  $\phi(y) < 1$  and  $y_i < 1$  for some  $i$ , then  $\phi(z_i, y_{-i}) < 1$  for every  $z_i < 1$ .

These four properties are satisfied, for example, when  $\phi$  is one of the functions

- $\phi_1(y) = \max\{y_1, y_2, \dots, y_{N-1}\}$ ,
- $\phi_2(y) = \min\{y_1, y_2, \dots, y_{N-1}\}$ ,
- $\phi_3(y) = 1/(N - 1) \sum_{j=1}^{N-1} y_j$ ,
- $\psi_k(y) = \{k\text{th highest value of } \{y_1, \dots, y_{N-1}\}\}$ ,  $k = 1, \dots, N - 1$ .

Diagonal games include many models of competition with complete information. For example, in one-unit auctions,  $\phi = \phi_1$ ; in  $k$ -unit auctions,  $\phi = \psi_k$ ; in wars of attrition, preemption, or Bertrand competition,  $\phi = \phi_2$ .

THEOREM 3.15. *Every quasiconcave diagonal game satisfying condition (C) below is approximately better-reply secure; thus it possesses an approximate equilibrium.*

(C) *There is  $\alpha \in ]0, \frac{1}{2}[$  such that for every  $x \in [0, 1]^N$  and  $i \in N$ , if  $x_i = \phi(x_{-i})$ , then there is  $\alpha_i(x) \in ]\alpha, 1 - \alpha[$  such that  $h_i(x) = \alpha_i(x)f_i(x_i, \phi(x_{-i})) + (1 - \alpha_i(x))g_i(x_i, \phi(x_{-i}))$ .*

Condition (C) means that  $h_i$  is a strict convex combination of  $g_i$  and  $f_i$  with weights that are bounded below. In auctions, for example, the winner is usually decided uniformly among highest bidders; thus the payoff of a player in the case of ties is a strict convex combination between his payoff if he wins,  $g_i$ , and if he loses,  $f_i$ . The coefficient of the convex combination depends on how many players are tied, inducing a discontinuity on  $h_i$ . The probability of being selected or not selected is bounded below by  $1/N = \alpha$ .

<sup>11</sup>For every  $(x, y) \in [0, 1]^{N-1} \times [0, 1]^{N-1}$ ,  $x \ll y$  (resp.  $x \leq y$ ) means  $x_i < y_i$  (resp.  $x_i \leq y_i$ ) for every  $i \in \{1, \dots, N - 1\}$ .

<sup>12</sup>We denote  $(z_i, y_{-i}) := (y_1, \dots, y_{i-1}, z_i, y_{i+1}, \dots, y_{N-1})$ .

Hereafter, we give a sketch of the proof of [Theorem 3.15](#). See [Appendix A.1](#) for a detailed proof.

Under assumption (C), the game is payoff secure. Consequently, if  $(x, v) \in \bar{\Gamma}$  is a Reny solution, then

$$\sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i, \quad i \in N.$$

To prove that  $x$  is an approximate equilibrium profile, one has to check four different cases: first, if  $x_i \neq \phi(x_{-i})$  for every  $i$ , then the payoff functions are continuous at  $x$ ,  $v = u(x)$ , and the equation above defining Reny solution implies that  $(x, v)$  is a Nash equilibrium. Second, if there exists  $i$  such that  $x_i = \phi(x_{-i}) \in ]0, 1[$ , then anonymity, representativity, and monotonicity give  $\phi(x_{-j}) \in ]0, 1[$  for every  $j$ . Then the marginal continuity property is satisfied at  $x$ , and from [Corollary 3.13](#),  $(x, v)$  is an approximate equilibrium. Third, assume there is a player  $i$  such that  $x_i = \phi(x_{-i}) = 0$ . By anonymity and monotonicity,  $\phi(x_{-j}) = 0$  for every  $j \in N$ . Let  $(x^n)_{n \in \mathbb{N}}$  be a sequence such that  $(x^n, u(x^n)) \rightarrow (x, v)$ . Define a sequence of profiles  $(y^n)_{n \in \mathbb{N}}$  as follows: let  $j \in N$  be any player. If  $v_j \leq f_j(0, 0)$  and  $x_j = 0$ , define  $y_j^n := 0$  for every  $n$ . Otherwise, define  $y_j^n := x_j^n$  for every  $n$ . We check that  $y^n$  is an  $\varepsilon^n$  equilibrium for some  $\varepsilon^n \rightarrow 0$ . In the last case, there is a player  $i \in N$  such that  $x_i = \phi(x_{-i}) = 1$ : this is similar to the third case.

#### 4. EXISTENCE OF SOLUTIONS FOR GAMES IN MIXED STRATEGIES

We let  $G$  be a metric compact game, let  $G'$  be its mixed extension, and let  $\bar{\Gamma}'$  be the closure of the graph of  $G'$ . As previously defined,  $M_i = \Delta(X_i)$  is the set of mixed strategies of player  $i$ . This is a compact Hausdorff metrizable set under the weak\* topology. Let  $M = \prod_i M_i$ .

A Reny solution (resp. an approximate equilibrium) of  $G'$  is called a mixed Reny solution (resp. approximate mixed equilibrium) of  $G$ . A mixed Reny solution always exists from [Theorem 3.3](#) because  $G'$  is compact and quasiconcave.

In the next subsection, we establish a formal link between the set of Reny solutions of  $G'$  and the set of mixed sharing rule equilibria (as defined by Simon and Zame originally; see [Definitions 2.13](#) and [2.17](#) above) by proving that the intersection of these sets is nonempty. Then we prove that if  $G'$  is approximately better-reply secure, some approximate equilibria can be obtained as limits of Nash equilibria of finite discretizations of the initial game. The approximation methodology is illustrated in auctions with correlated types or values.

##### 4.1 Linking approximate and Simon–Zame equilibria with Reny solutions

The intersection of mixed Reny solutions and mixed sharing rule equilibria permits us to localize mixed approximate equilibria, as the following theorem proves (see the proof in [Appendix A.2](#)).

**THEOREM 4.1.** *Mixed approximate equilibria are always in the intersection of mixed Reny solutions and mixed sharing rule equilibria.*<sup>13</sup>

Importantly, this intersection is always nonempty, which is a consequence of a simple limit argument we now explain. We let  $\mathcal{D}_0$  be the set of all finite subsets  $\prod_{i \in N} D_i$  of  $M$ . Consider the inclusion relationship on  $\mathcal{D}_0$ : it is reflexive, transitive, and binary. Then each pair  $\prod_{i \in N} D_i$  and  $\prod_{i \in N} D'_i$  in  $\mathcal{D}_0$  has an upper bound  $\prod_{i \in N} (D_i \cup D'_i)$  in  $\mathcal{D}_0$ . The pair  $(\mathcal{D}_0, \subset)$  is called a directed set. To every  $D = \prod_{i \in N} D_i \in \mathcal{D}_0$ , we can associate  $(m^D, u(m^D))$ , where  $m^D$  is a mixed Nash equilibrium of the finite game restricted to  $D$ . This defines a mapping<sup>14</sup> from  $\mathcal{D}_0$  to  $\overline{\Gamma}$ , called a net (of  $\overline{\Gamma}$ ). A limit point  $(m, v) \in \overline{\Gamma}$  of this net, denoted  $(m^D, u(m^D))_{D \in \mathcal{D}_0}$ , is defined by the following property: for every neighborhood  $V_{m,v}$  of  $(m, v)$  and every  $D = \prod_{i \in N} D_i \in \mathcal{D}_0$ , there exists  $D' \in \mathcal{D}_0$  with  $D \subset D'$  such that  $(m^{D'}, u(m^{D'})) \in V_{m,v}$ .

**DEFINITION 4.2.** A pair  $(m, v) \in \overline{\Gamma}$  is a limit equilibrium of  $G'$  if it is a limit point of a net  $(m^D, u(m^D))_{D \in \mathcal{D}_0}$  of mixed Nash equilibria of the finite game restricted to  $D$ .

**THEOREM 4.3.** *Every compact metric game has a limit equilibrium. Every limit equilibrium is a mixed Reny solution and is a mixed sharing rule equilibrium. Consequently, the intersection between mixed Reny solutions and mixed sharing rule equilibria is nonempty, and if  $G'$  is better-reply secure, then every limit equilibrium is a mixed Nash equilibrium.*

The existence of a limit equilibrium is obvious and is a consequence of the compactness of  $\overline{\Gamma}$ . The rest of the proof is presented in the [Appendix](#) and is an adaptation of the arguments of Simon and Zame.

**REMARK 4.4.** Observe that this provides a very short and constructive proof of Reny's existence result for games in mixed strategies ([Reny 1999](#)) because if  $G'$  is better-reply secure, every limit equilibrium is a mixed Nash equilibrium.

#### 4.2 Weak strategic approximation

The idea of using a sequence of finite games to detect Nash equilibria goes back to [Dasgupta and Maskin \(1986a\)](#). This has been formalized by [Reny \(2011\)](#) in the class of better-reply secure games via the notion of strategic approximation. We can extend this method to approximately better-reply secure games.

<sup>13</sup>From [Section 3](#), this is also true in pure strategies. Nevertheless, it cannot be directly proved by applying [Section 3](#) to  $G'$ , simply because a pure sharing rule equilibrium of  $G'$  may not be a mixed sharing rule equilibrium of  $G$ .

<sup>14</sup>By definition,  $m^D = (m_i^D)_{i \in N}$  is an element of  $\prod_{i \in N} \Delta(\Delta(X_i))$ . More precisely, it is a profile of probability measures on finite subsets of  $\Delta(X_i)$ , where  $i \in N$ . Given  $i \in N$ , let  $\{\sigma_1, \dots, \sigma_K\}$  be the support of  $m_i^D$  and let  $p_1, \dots, p_K$  be the associated weights. By abuse of notation, we can define  $m_i^D = \sum_{k=1}^K p_k \cdot \sigma_k$ , which is now an element of  $\Delta(X_i)$ . Up to this identification,  $(m^D, u(m^D))$  can be seen as an element of  $\overline{\Gamma}$ .

**DEFINITION 4.5.** A game  $G$  admits a weak strategic approximation if there is a sequence of finite sets  $D_n \subset M$  such that all accumulation points of mixed Nash equilibria of the game restricted to  $D_n$  are approximate equilibria of  $G'$ .

**THEOREM 4.6.** *If a compact metric game  $G'$  is approximately better-reply secure, it has a weak strategic approximation. Moreover, if a compact metric game admits a weak strategic approximation, it has an approximate equilibrium.*

The proof of the first part (in the [Appendix](#)) is an adaptation of Reny's arguments ([Reny 2011](#)), thanks to the notion of limit equilibrium. The second part is straightforward from the existence of accumulation points for every sequence in a compact set.

**REMARK 4.7.** A condition that guarantees the existence of a weak strategic approximation was given by [Blackwell and Girshick \(1954\)](#) (see [Theorem 2.3.3](#)) in zero-sum two-player games  $G = (X_1, X_2, u_1, u_2)$ . It could be stated as follows: for every  $\epsilon > 0$ , there exists a finite set  $D_1 \subset X_1$  such that, for every  $x_1 \in X_1$ , there exists a mixture  $\sigma_1$  of elements of  $D_1$  such that  $u_1(\sigma_1, x_2) \geq u_1(x_1, x_2) - \epsilon$  for every  $x_2 \in X_2$ . Assume this is true. From the boundedness of payoffs, there exists a finite subset  $D_2 \subset X_2$  such that for every  $x_2 \in X_2$ , there is  $d_2 \in D_2$  with,  $\forall d_1 \in D_1, |u_2(d_1, x_2) - u_2(d_1, d_2)| \leq \epsilon$ . Then every mixed Nash equilibrium of the restriction of  $G$  to  $D_1 \times D_2$  is an  $\epsilon$ -Nash equilibrium of  $G$  (see [Blackwell and Girshick 1954](#) for more details). Thus, this game admits a weak strategic approximation.

The following diagonal game does not satisfy the assumption of Blackwell and Girshick, but satisfies the assumption of [Theorem 4.8](#) below (and thus admits a weak strategic approximation): consider  $X_1 = X_2 = [-1, 1]$ ,  $u_1(x_1, x_2) = 0$  if  $x_1 \neq x_2$ ,  $u_1(x_1, x_2) = x_1$  otherwise, and  $u_2 = -u_1$ . If there exists a finite subset  $D_1 \subset X_1$  satisfying the Blackwell and Girshick condition, taking  $x_2 \in (0, 1]$ , which is not in  $D_1$  and  $x_1 = x_2$ , we get a contradiction for  $\epsilon > 0$  small enough.

### 4.3 Applications

#### *Non-quasiconcave two-player diagonal games*

**THEOREM 4.8.** *Every two-player diagonal game in which  $h$  is continuous admits a weak strategic approximation (and thus an approximate equilibrium in mixed strategies).*

The proof of [Theorem 4.8](#) consists of constructing a weak strategic approximation (see [Appendix A.5](#)). Interestingly, the approximation is endogenous (i.e., game dependent). The multi-player case is investigated in the next application.

**EXAMPLE 4.9.** Sion and Wolfe's zero-sum game shows that [Theorem 4.8](#) can be false for games with two lines of discontinuities instead of one (see [Sion and Wolfe 1957](#)).  $\diamond$

Some particular cases covered by [Theorem 4.8](#) follow.



EXAMPLE 4.10 (Bertrand duopoly with discontinuous costs). [Hoernig \(2007\)](#) introduced the following modification of Bertrand's game: each firm  $i = 1, 2$  chooses a price  $p_i \in [0, 1]$ ; the demand is  $D(p_1, p_2) = \max\{0, 1 - \min(p_1, p_2)\}$ ; the total (symmetric) cost for each firm is  $C(q) = \tilde{C} \in (0, \frac{1}{4})$  if the production  $q$  is positive, and  $C(0) = 0$  otherwise. Assuming equal sharing at ties, [Hoernig \(2007\)](#) proved that the game has no mixed Nash equilibrium. By [Theorem 4.8](#), it has an approximate equilibrium.  $\diamond$

EXAMPLE 4.11 (Bertrand–Edgeworth duopoly with capacity constraints). There are two firms. Firm  $i$  has an endowment of  $C_i$  units of the commodity (the capacity of a zero-cost technology). Firms choose prices ( $p_1$  and  $p_2$ ). The firm choosing the lowest price (say  $p$ ) serves the entire market  $D(p)$  up to its capacity. The residual demand  $D(p) - C_i$  is met by the other firm (up to its capacity as well). If the duopolists set the same price, they share the market according to some rule  $h$ . If  $h$  shares the market in proportion to the capacities, [Dasgupta and Maskin \(1986b\)](#) proved the existence of a mixed equilibrium. [Theorem 4.8](#) proves the existence of an approximate equilibrium for every continuous  $h$ .  $\diamond$

*Bayesian diagonal games and auctions with correlated types* In many economic models, such as auctions, players do not have full knowledge about other player's evaluations. This leads naturally to the following class of Bayesian diagonal games. At stage 0, a type  $t = (t_1, \dots, t_N) \in T = T_1 \times \dots \times T_N$  is drawn according to some joint probability distribution  $p$ , and each player  $i$  is informed of his own type  $t_i$  (correlations between types are allowed). At stage 1, each player  $i$  is asked to choose an element  $x_i \in [0, 1]$  (interpreted as a bid). The payoff of player  $i$  is

$$u_i(t, x_i, x_{-i}) = \begin{cases} f_i(t, x_i, \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) > x_i \\ g_i(t, x_i, \phi_i(x_{-i})) & \text{if } \phi_i(x_{-i}) < x_i \\ h_i(t, x) & \text{if } \phi_i(x_{-i}) = x_i, \end{cases}$$

where  $f_i(t, \cdot, \cdot)$  and  $g_i(t, \cdot, \cdot)$  are two continuous mappings on  $[0, 1] \times [0, 1]$ , and  $\phi_i : \mathbb{R}^{N-1} \rightarrow [0, 1]$  is a monotone aggregation function (see [Application 3.14](#)). The mapping  $h$  is the tie-breaking rule and may be discontinuous (but is measurable). Actually, in first-price auctions, for example, the tie-breaking rule may depend on how many players offer the highest bid. That is why in our definition,  $h_i$  may be discontinuous and have more arguments than  $f_i$  and  $g_i$ . The game is called a game of private values if for every  $i$ ,  $u_i$  depends only on its own type  $t_i$  and does not depend on  $t_{-i}$ .

To avoid additional notation and measurability issues, we assume the type space  $T$  to be finite. This assumption could be relaxed (see [Remark A.2](#) in [Appendix A.6](#)).

**THEOREM 4.12.** *Every Bayesian diagonal game admits a weak strategic approximation (and so a mixed approximate equilibrium)<sup>15</sup> if for every  $i \in N$  and  $t \in T$ , one has the following conditions:*

<sup>15</sup>Importantly, for every  $\varepsilon > 0$ , the  $\varepsilon$  equilibria we build in the proof are tie-breaking-rule independent.

(a) We have  $f_i(t, 0, 0) \leq h_i(t, 0, \dots, 0) \leq g_i(t, 0, 0)$ .

(b1) We have  $f_i(t, 1, 1) \geq h_i(t, 1, \dots, 1) \geq g_i(t, 1, 1)$  or<sup>16</sup> (b2) there is  $\eta > 0$  such that there is always a best response of each type in  $[0, 1 - \eta]$ .

(c1) There are only two players or (c2) values are private.

EXAMPLE 4.13 (One unit first-pay, second-pay, and all-pay auctions). Take any  $N$ -player auction where the winner is the player with the maximal bid. More precisely, suppose that player  $i$ 's value for the object is  $v_i(t) \in ]0, 1[$ . If  $x_i > \bar{x}_{-i} := \max_{j \neq i} x_j$ ,  $i$  wins the auction and pays a price  $p_i(x_i, \bar{x}_{-i}) \geq 0$ . His final payoff is then  $g_i(t, x_i, \bar{x}_{-i}) = v_i(t) - p_i(x_i, \bar{x}_{-i})$ , where  $p_i$  is continuous and nondecreasing in both variables, and  $p_i(y, y) = y$  for every  $y$ . If  $x_i < \bar{x}_{-i}$ , player  $i$  loses the auction and pays a transfer  $\tau_i(x_i) \geq 0$ . His payoff is then  $f_i(t, x_i, \bar{x}_{-i}) = -\tau_i(x_i)$ , where  $\tau_i(x_i)$  is continuous and nondecreasing, and  $\tau_i(0) = 0$ . In case of a tie ( $x_i = \bar{x}_{-i}$ ), the winner is selected uniformly among the set of players with maximum bid. For example, in first-price and second-price auctions,  $\tau_i = 0$ . In all-pay auctions,  $\tau_i = -x_i$ . In first-pay and all-pay auctions,  $p_i(x_i, \bar{x}_{-i}) = \max\{x_i, \bar{x}_{-i}\}$ ; in second-price auctions,  $p_i(x_i, \bar{x}_{-i}) = \bar{x}_{-i}$ . In this general model,  $0 = f_i(t, 0, 0) < g_i(t, 0, 0) = v_i(t)$  and  $0 \leq h_i(t, x) \leq v_i(t)$ , so that condition (a) is satisfied. Condition (b1) is satisfied in first-price and second-price auctions, but not in all-pay auctions. However, condition (b2) is satisfied in these three types of auctions because player  $i$  always has a best response in  $[0, \max_x v_i(t) + \varepsilon]$ . Thus Theorem 4.12 applies when there are two players or values are private.  $\diamond$

EXAMPLE 4.14 (Multi-unit auctions). Consider the previous model with the following modification:  $K$  homogeneous units of an indivisible good are sold, but each bidder  $i = 1, \dots, N$  ( $N \geq K$ ) can buy only one unit of the good. Player  $i$  wins if his bids is among the  $K$  highest bids. In case of a tie, the remaining winners are chosen at random among the tied players. Theorem 4.12 applies, and here  $\phi(x_1, \dots, x_{N-1})$  is simply the  $K$ th highest of  $x_1, \dots, x_{N-1}$ .  $\diamond$

EXAMPLE 4.15 (Double auction). Suppose player 1 is a buyer with a value  $v(t_1, t_2) \in ]0, 1[$  and player 2 is a seller with a cost  $c(t_1, t_2) \in ]0, 1[$ . Player 1 chooses a maximum bid  $x_1 \in [0, 1]$  and player 2 chooses a minimum price  $x_2$ . If  $x_1 < x_2$ , there is no trade (so that  $f_1(t, x_1, x_2) = g_2(t, x_1, x_2) = 0$ ). If  $x_1 \geq x_2$ , there is a trade at price  $p = (x_1 + x_2)/2$ , so that  $h_1(t, x_1, x_2) = g_1(t, x_1, x_2) = v(t_1, t_2) - (x_1 + x_2)/2$  and  $h_2(t, x_1, x_2) = f_2(t, x_1, x_2) = (x_1 + x_2)/2 - c(t_1, t_2)$ . Consequently,  $f_1(t, 0, 0) = 0 < h_1(t, 0, 0) = g_1(t, 0, 0) = v(t)$  and  $f_2(t, 0, 0) = -c(t) = h_2(t, 0, 0) < g_2(t, 0, 0) = 0$ : condition (a) is satisfied. Condition (b1) is satisfied similarly, but condition (b2) is not satisfied for the seller. Since the game has only two players, Theorem 4.12 applies.  $\diamond$

<sup>16</sup>Conditions (b1) and (b2) are boundary conditions at 1. Condition (b1) is satisfied by first-price, second-price, multi-unit, and double auctions, but not for all-pay auctions. Condition (b2) is true for all-pay, first-price, second-price, and multi-unit auctions, but not for double auctions (see Examples 4.13, 4.14, and 4.15).

An open question is whether [Theorem 4.12](#) holds for  $N$ -player diagonal games without condition (c2). But without conditions (a) or (b), approximate equilibria may not exist as the following zero-sum example shows.

**EXAMPLE 4.16.** Consider a zero-sum timing game, viewed as a diagonal game with constant payoff functions  $f$ ,  $g$ , and  $h$ . Each player should decide when to stop the game between 0 and 1. The game stops at the first moment when one of the two players stops. If both players stop simultaneously before the exit time  $t = 1$  or no player stops before time  $t = 1$ , then there is a tie (payoff is given by  $h$ ). Player 2 has two types  $A$  and  $B$  with equal probabilities. Player 1 has only one type. If player 1 stops first, he gets  $f = 1$ . If player 1 stops second he gets  $g = -1$ . The payoffs depend on the type of player 2 only when the players stop simultaneously. If his type is  $A$ , player 1 has an advantage and gets the payoff  $h = 3$ , and if his type is  $B$ , player 1 has a disadvantage and gets the payoff  $h = -2$ . We can prove that  $\max \min \leq -\frac{1}{2}$  and that  $\min \max \geq -\frac{1}{4}$ , so that the game has no value and so no approximate equilibrium (see [Appendix A.7](#)).  $\diamond$

**REMARK 4.17.** [Theorem 4.12](#) is to be compared with the one in [Jackson and Swinkels \(2005\)](#). They show the existence of a Nash equilibrium that is tie-breaking-rule independent in multi-unit auctions with private values and uncorrelated types. Recalling that when types are correlated, a Nash equilibrium may not exist ([Fang and Morris 2006](#)), the existence of an approximate equilibrium is the best one can hope.

**REMARK 4.18.** The existence of Nash equilibria for Bayesian games is well understood for Bayesian games with continuous payoffs ([Balder 1988](#), [Milgrom and Weber 1985](#)). Recently, it has been extended to discontinuous payoffs under a uniform payoff security assumption (e.g., [Carbonell-Nicolau and McLean 2015](#)), with application to auctions.

## 5. CONCLUSION

Our paper proposes a unifying framework to study the existence of approximate and sharing rule equilibria in discontinuous games, which links Simon–Zame and Reny approaches in pure and mixed strategies.

In the first part, we focus on quasiconcave compact games in *pure strategies*. A new relaxation of Nash equilibrium (the Reny solution) is shown to always exist. It provides tight conditions, in the spirit of Reny’s conditions, that guarantee the existence of an approximate equilibrium. Reny’s solution is also used to solve an open problem in [Jackson et al. \(2002\)](#), namely the existence of a sharing rule equilibrium in pure strategies (up to now, existence was known only for games in mixed strategies). As applications, we prove the existence of sharing rule equilibria in many economic models with discontinuous preferences and approximate equilibria in a large class of diagonal games.

In the second part, we concentrate on metric compact games in *mixed strategies*. We prove that the intersection between the sets of Simon–Zame and Reny solutions contains the set of approximate equilibria. Moreover, this intersection is nonempty. This shows that the three main solution concepts discussed in this paper are strongly connected. As an application, we prove the existence of an approximate equilibrium in a large class of auctions with interdependent types and values.

## APPENDIX

## A.1 Proof of Theorem 3.15

Under assumption (C), the game is payoff secure: indeed, if  $x_i \neq \phi(x_{-i})$ ,  $u_i$  is continuous at  $x$ ; thus  $x_i$  is secure for player  $i$ . If  $x_i = \phi(x_{-i})$ , then player  $i$  can secure his payoff (up to an arbitrary  $\varepsilon > 0$ ), increasing or decreasing  $x_i$  slightly or keeping it constant. Consequently, if  $(x, v) \in \bar{\Gamma}$  is a Reny solution of the game, then

$$\sup_{d_i \in [0,1]} \underline{u}_i(d_i, x_{-i}) = \sup_{d_i \in [0,1]} u_i(d_i, x_{-i}) \leq v_i \quad \forall i \in N. \quad (2)$$

Now, we prove that  $x$  is an approximate equilibrium profile by checking four different cases. In the *first case*, assume for every player  $i \in N$ ,  $x_i \neq \phi(x_{-i})$ . Thus, payoffs are continuous at  $x$  and  $v = u(x)$ . From (2),  $(x, v) = (x, u(x))$  is a Nash equilibrium. In the *second case*, there exists a player  $i$  such that  $x_i = \phi(x_{-i}) \in ]0, 1[$ . Then  $\phi(x_{-j}) \in ]0, 1[$  for every  $j$ . Indeed, for every  $j \neq i$ , either  $x_j \geq x_i > 0$ , and anonymity and representativity imply that  $\phi(x_{-j}) > 0$  and monotonicity implies that  $\phi(x_{-j}) \leq \phi(x_{-i}) < 1$ , or  $x_j \leq x_i < 1$  and we get similarly  $\phi(x_{-j}) \in ]0, 1[$ . Thus, the marginal continuity property is satisfied at  $x$ , since for every player  $j \in N$ , we have

$$\sup_{d_j \in [0,1]} u_j(d_j, x_{-j}) = \max \left\{ \sup_{d_j < \phi(x_{-j})} f_j(d_j, \phi(x_{-j})), \sup_{d_j > \phi(x_{-j})} g_j(d_j, \phi(x_{-j})) \right\}$$

from assumption (C) and from continuity of  $\phi$ ,  $f_j$ , and  $g_j$ . Thus, Corollary 3.13 implies that  $(x, v)$  is an approximate equilibrium. In the *third case*, there exists a player  $i$  such that  $x_i = \phi(x_{-i}) = 0$ . Then  $\phi(x_{-j}) = 0$  for every player  $j$ : indeed, anonymity gives  $\phi(x_{-j}) = 0$  for every  $j$  such that  $x_j = 0$ , and monotonicity gives  $\phi(x_{-j}) = 0$  for every  $j$  such that  $x_j > 0$ . Now let  $(x^n)_{n \in \mathbb{N}}$  be a sequence of profiles such that  $(x^n, u(x^n)) \rightarrow (x, v)$ . For every player  $j$  such that  $x_j = 0$  and  $v_j \leq f_j(0, 0)$ , we let  $y_j^n := 0$  for every integer  $n$  and let  $(y_j^n)_{n \in \mathbb{N}} := (x_j^n)_{n \in \mathbb{N}}$  otherwise. This defines a sequence of profiles  $(y^n)_{n \in \mathbb{N}}$  converging to  $x$ . Let us prove that  $y^n$  is an  $\varepsilon^n$  equilibrium for some  $\varepsilon^n \rightarrow 0$ . From continuity of  $\phi$ ,  $\lim_{n \rightarrow +\infty} \phi(y_{-j}^n) = \phi(x_{-j}) = 0$  for every player  $j$ . In particular, from (2), and since  $g_j$  is continuous, we get

$$\sup_{d_j \in [0,1]} g_j(d_j, 0) \leq v_j. \quad (3)$$

Fix some player  $j \in N$ , and consider three subcases:

- First, assume  $f_j(0, 0) = g_j(0, 0)$ . In this case, from assumption (C),  $f_j(0, 0) = g_j(0, 0) = h(0, x_{-j})$ , and thus  $d_j \rightarrow u_j(d_j, x_{-j})$  is continuous. In particular,  $v_j = u_j(x)$  and (2) implies that player  $j$  plays optimally at  $x_j$ . By continuity, player  $j$  is  $\varepsilon_n$ -optimizing by playing  $y_j^n$  against  $y_{-j}^n$  (because  $y^n$  converges to  $x$ ) for some sequence of positive reals  $(\varepsilon_n)_{n \in \mathbb{N}}$  converging to 0.
- Second, assume  $v_j \leq f_j(0, 0)$  and  $f_j(0, 0) \neq g_j(0, 0)$ . From (3),  $g_j(0, 0) \leq v_j \leq f_j(0, 0)$ , and since  $f_j(0, 0) \neq g_j(0, 0)$ , we get  $f_j(0, 0) > g_j(0, 0)$ . But this implies that  $g_j(\cdot, 0)$  is nonincreasing on  $[0, 1]$ : indeed, otherwise, for  $\varepsilon > 0$  small enough, we

would get a contradiction with the quasiconcavity of  $y_j \rightarrow u_j(y_j, \varepsilon, \dots, \varepsilon)$  and the inequality  $u_j(0, \varepsilon, \dots, \varepsilon) = f_j(0, \varepsilon) > u_j(2\varepsilon, \varepsilon, \dots, \varepsilon) = g_j(2\varepsilon, \varepsilon)$ .

Consequently, we finally get

$$\sup_{d_j \in [0,1]} g_j(d_j, 0) = g_j(0, 0). \tag{4}$$

Moreover, since  $u_j(0, x_{-j}) = h_j(0, x_{-j}) \in ]g_j(0, 0), f_j(0, 0)[$  (because of assumption (C)), the fact that  $g_j(\cdot, 0)$  is nonincreasing on  $[0, 1]$  also implies that  $x_j = 0$  (otherwise  $g_j(0, 0) < u_j(0, x_{-j}) \leq v_j = u_j(x) = g_j(x_j, 0)$  and (4) would be false). In particular, by definition, this implies  $y_j^n = 0$  for every  $n$ .

From  $g_j(0, 0) < f_j(0, 0)$ , assumption (C), the continuity of  $g_j$  and  $f_j$ , (4), and the fact that  $\phi(y_{-j}^n)$  converges to 0, there exists a real sequence  $\varepsilon_n \rightarrow 0$  such that one has

$$g_j(\phi(y_{-j}^n), \phi(y_{-j}^n)) - \varepsilon_n \leq \sup_{d_j \in [0, \phi(y_{-j}^n)]} f_j(d_j, \phi(y_{-j}^n)) - \varepsilon_n \leq f_j(0, \phi(y_{-j}^n)) \tag{5}$$

$$\sup_{d_j \in [0,1]} g_j(d_j, \phi(y_{-j}^n)) - \varepsilon_n \leq g_j(\phi(y_{-j}^n), \phi(y_{-j}^n)) \leq h_j(\phi(y_{-j}^n), y_{-j}^n) \tag{6}$$

and

$$g_j(\phi(y_{-j}^n), \phi(y_{-j}^n)) < h_j(\phi(y_{-j}^n), y_{-j}^n) < f_j(\phi(y_{-j}^n), \phi(y_{-j}^n)), \tag{7}$$

the last equation being true because the inequality  $g_j(0, 0) < f_j(0, 0)$  remains true on a neighborhood of  $(0, 0)$ , and from assumption (C).

This implies that player  $j$  is  $2\varepsilon_n$ -optimizing by playing  $y_j^n = 0$  against  $y_{-j}^n$ : indeed if  $\phi(y_{-j}^n) > 0$ , this is a consequence of (5), (6), and (7); if  $\phi(y_{-j}^n) = 0$ , this is a consequence of (6).

- Third, assume  $f_j(0, 0) \neq g_j(0, 0)$  and  $v_j > f_j(0, 0)$  (thus  $y_j^n = x_j^n$  for every  $n$ ). From assumption (C) and the definition of  $v_j$ , we deduce that  $f_j(0, 0) < h_j(0, x_{-j}) < g_j(0, 0)$ , but from (3),  $g_j(0, 0) \leq v_j$ , and we finally get

$$f_j(0, 0) < h_j(0, x_{-j}) < g_j(0, 0) = v_j. \tag{8}$$

Now, first assume  $x_j = 0$ . Thus, we should have  $u_j(x^n) = g_j(x_j^n, \phi(x_{-j}^n))$  for  $n$  large enough, which requires  $x_j^n > \phi(x_{-j}^n)$  for  $n$  large enough. By monotonicity of  $\phi$  and since  $x_i^n \geq y_i^n$  for every player  $i$ , we get  $y_j^n = x_j^n > \phi(x_{-j}^n) \geq \phi(y_{-j}^n)$ . Thus,  $u_j(y^n) = g_j(y_j^n, \phi(y_{-j}^n))$  and by continuity of  $g_j$ , this converges to  $g_j(0, 0) = v_j$ , when  $n$  tends to infinity.

From (3) and (8), we get

$$\max\left\{f_j(0, 0), h_j(0, x_{-j}), \sup_{d_j \in [0,1]} g_j(d_j, 0)\right\} \leq g_j(0, 0). \tag{9}$$

Since  $u_j(y^n)$  converges to  $g_j(0, 0)$ , we get that  $y_j^n$  is an  $\varepsilon'_n$ -best response against  $y_{-j}^n$  for some sequence of positive reals  $(\varepsilon'_n)_{n \in \mathbb{N}}$  converging to 0 (use continuity of  $f_j$ ,  $g_j$  and assumption (C)).

Second, assume  $x_j > 0$ . Since  $\phi(x_{-j}) = 0$ ,  $u_j$  is continuous at  $x$  and  $v_j = g_j(x_j, 0) = u_j(x)$ . From (2),  $x_j$  is a best response against  $x_{-j}$ ; thus we finally have

$$\max \left\{ f_j(0, 0), h_j(0, x_{-j}), \sup_{d_j \in [0, 1]} g_j(d_j, 0) \right\} \leq u_j(x) \tag{10}$$

and, similarly to the previous case, from the continuity of  $f_j$  and  $g_j$ , assumption (C), and the continuity of  $u_j$  around  $x$ , we obtain that  $y_j^n = x_j^n$  is an  $\varepsilon'_n$ -best response against  $y_{-j}^n$  for some sequence of positive reals  $(\varepsilon'_n)_{n \in \mathbb{N}}$  converging to 0.

This ends the three subcases, and  $x$  is an approximate equilibrium profile associated to the sequence  $(y^n)_{n \in \mathbb{N}}$ .

In the *last case*, we assume there exists  $i \in N$  such that  $x_i = \phi(x_{-i}) = 1$ : this can be proved as in the third case.

### A.2 Proof of Theorem 4.1

The first inclusion is a consequence of Theorem 3.10 applied to  $G'$ .

For the second inclusion, consider a sequence  $(m^n, u(m^n))_{n \in \mathbb{N}^*}$  of  $1/n$ -mixed Nash equilibria converging to  $(m, v)$ . To prove that it is a mixed sharing rule equilibrium, we show how to adapt Simon and Zame's proof.

Note that the sequence  $(m^n)_{n \in \mathbb{N}}$  converges (weakly) to  $m$ . We let  $E$  be the space of  $\mathbb{R}^N$ -valued vector measures on  $X$ , endowed with the weak\* topology. Consider the sequence  $(u \cdot m^n)_{n \in \mathbb{N}}$  of the compact space  $E$  (here  $u \cdot m^n$  denotes the  $\mathbb{R}^N$ -valued measure on  $X$  defined by  $u \cdot m^n(F) = \int_F u dm^n$  for every Borelian set  $F$  of  $X$ ). Without any loss of generality, up to a subsequence, this sequence converges to some measure  $\nu$ . From Lemma 2 of Simon and Zame (1990, p. 867), there exists a Borel measurable selection  $q$  of  $Q$ , the multi-valued function from  $X$  to  $\mathbb{R}^N$ , defined by  $Q(x) = \text{co} \overline{\Gamma}_x$ , such that  $\nu = q \cdot m$  (remark that the proof of this lemma does not use the support of  $m^n$ , but only the fact that  $u$  is a selection of  $Q$ ). Define, for every player  $i$  and every integer  $k > 0$ ,  $H_i^k = \{x_i \in X_i : \int q_i d(\delta_{x_i} \times m_{-i}) > \int q_i d(m_i \times m_{-i}) + 1/k\}$ . Let us prove that  $m_i(H_i^k) = 0$  for every integer  $k > 0$ .

Otherwise, consider  $K \subset H_i^k \subset U$ , where  $K$  is compact,  $U$  is open,  $m_i(U - K) < \varepsilon$  with  $\varepsilon > 0$ , and with  $m_i(K) > 0$ . We let  $f : X_i \rightarrow [0, 1]$  be a continuous function that is identically equal to 1 on  $K$  and 0 on the complement of  $U$ . Consider the strategy  $\beta_i^n = f m_i^n / \int f dm_i^n$ . Then  $u_i(\beta_i^n, m_{-i}^n) > u_i(m^n) + 1/(2k)$  for  $n$  large enough and  $\varepsilon > 0$  small enough (see Step 4 in Simon and Zame), which contradicts the fact that  $m^n$  is a  $1/n$ -Nash equilibrium (for  $n$  large enough). Thus  $m_i(H_i^k) = 0$  for every integer  $k > 0$ . Now let  $H_i = \{x_i \in X_i : \int q_i d(\delta_{x_i} \times m_{-i}) > \int q_i d(m_i \times m_{-i})\}$ . Since  $H_i = \bigcup_k H_i^k$ , we get  $m_i(H_i) = 0$ .

From Step 5 in Simon and Zame, there exists a modification  $\tilde{q}$  of  $q$ , such that  $q = \tilde{q}$  except on a set of  $m$  measure 0, such that  $m$  is a Nash equilibrium of the game  $\tilde{G} = ((X_i)_{i \in N}, (\tilde{q}_i)_{i \in N})$ , and  $\tilde{q}(m) = q(m)$ . More precisely, take  $\tilde{p}^i$  a Borel measurable selection from  $Q$  that minimizes the  $i$ th component of  $Q$ , define  $T = \{x \in X : x_i \in H_i \text{ for at least two indices } i \in N\}$ , define  $\tilde{q}(x) = \tilde{p}^i(x)$  if  $x \in H_i \times X_{-i}$  but  $x \notin T$ , and

$\tilde{q}(x) = q(x)$  otherwise. To prove that  $m$  is a Nash equilibrium of  $\tilde{G}$ , we follow Simon and Zame. By contradiction, assume that some player  $i$  has a profitable deviation  $x_i$ , that is,  $\tilde{q}_i(x_i, m_{-i}) > \tilde{q}_i(m) + \eta$  for some  $\eta > 0$ . Then the case  $x_i \notin H_i$  is not possible by definition of  $H_i$  (see Step 6, Case 1 in Simon and Zame). The case  $x_i \in H_i$  is as in Step 6, Case 2 in Simon and Zame, and implies that  $u_i(x_i, m_{-i}^n) > u_i(m^n) + \eta/2$  for  $n$  large enough, a contradiction.

### A.3 Proof of Theorem 4.3

A limit equilibrium  $(m, v)$  exists by compactness of  $\overline{\Gamma}$ . First let us prove that it is a Reny solution of  $\Gamma'$ . Fix  $d \in M$ . The definition of a limit equilibrium implies that for every neighborhood  $V$  of  $(m, v)$ , there exists  $m^V \in M$  and  $D'$  a finite subset of  $M$  containing  $d$  such that (i)  $\forall i \in N, \forall d' \in D', u_i(d', m_{-i}^V) \leq u_i(m^V)$  and (ii)  $(m^V, u(m^V)) \in V$ . Shrinking  $V$  to  $(m, v)$  implies that  $u_i(d, m_{-i}) \leq v_i$ , and since this is true for every  $d \in M$ ,  $(m, v)$  is a Reny solution of  $\Gamma'$ .

Now let us prove that  $(m, v)$  induces a solution à la Simon and Zame. Since  $M$  is a compact metric set, there exists a countable decreasing basis of neighborhoods  $V^n$  of  $(m, v)$  in  $\overline{\Gamma}$ . Since  $X$  is a compact metric set, there exists a sequence  $D^n = \prod_{i \in N} D_i^n$  of finite sets of pure strategies converging to  $X$  for the Hausdorff distance. By definition of a limit equilibrium (Definition 4.2), for every integer  $n$ , there exists a sequence of finite sets  $D'^n = \prod_{i \in N} D_i'^n$  of mixed strategies containing  $D^n$ , and a probability  $m^n$ , which is a Nash equilibrium of the game restricted to  $D'^n$  such that  $(m^n, u(m^n)) \in V^n$ . Recall that Simon and Zame's existence proof consists in approximating the game by a finite game in pure strategies (here  $D^n$ ), and in considering a weak limit of a sequence  $(m^n)_{n \in \mathbb{N}}$  of Nash equilibria of this approximation. We cannot apply Simon and Zame's proof directly to the Nash equilibria  $m^n$  of the finite games  $D'^n$ , because  $D'^n$  may contain mixed strategies. But  $D'^n \supset D^n$ : thus, no player  $i$  has a profitable deviation in  $D_i'^n$  against  $m^n$ , and we shall prove that this property is sufficient to adapt Simon and Zame's proof.

Let  $\tilde{G}$ ,  $\tilde{q}$ , and  $H_i$  be defined as in the proof of Theorem 4.1 (Section A.2 just above). To prove that  $m$  is a Nash equilibrium of  $\tilde{G}$ , assume that some player  $i$  has a profitable deviation  $x_i$  to  $m_{-i}$  that is,  $\tilde{q}_i(x_i, m_{-i}) > \tilde{q}_i(m)$ . Then the case  $x_i \notin H_i$  yields a contradiction (Step 6, Case 1 in Simon and Zame). For the second case, simply consider a sequence  $x_i^n$  converging to  $x_i$  such that  $x_i^n \in D_i^n$  (here, we use that  $D^n = \prod_{i \in N} D_i^n$  converges to  $X$  for the Hausdorff distance). A limit argument (Step 6, Case 2 in Simon and Zame) proves that  $u_i(x_i^n, m_{-i}^n) > u_i(m^n)$  for  $n$  large enough, a contradiction, because  $m^n$  is a Nash of  $D'^n$  and because  $x_i^n \in D_i^n \subset D_i'^n$ .

### A.4 Proof of Theorem 4.6

Let  $V \subset \overline{\Gamma}$  be the set of non-Reny solutions of  $G'$ . By definition, for every  $(m, v) \in V$ , there exists  $V^{(m,v)}$  (a neighborhood of  $(m, v)$ ),  $d^{(m,v)} \in M$ , and a player  $i \in N$  such that  $u_i(d_i^{(m,v)}, m_{-i}') > v_i'$  for every  $(m', v') \in V^{(m,v)}$ . This yields a collection of pairs  $(V^{(m,v)}, d^{(m,v)})$  for every  $(m, v) \in V$ . Since the set of Reny solutions is compact in  $\overline{\Gamma}$ ,  $V$  is open in  $\overline{\Gamma}$ ; thus there is a countable family of pairs  $\{(V^n, d^n)\}_n$  defined as above

such that  $V = \bigcup_{n \geq 0} V^n$ . Define  $D_i^n = \{d_i^1, \dots, d_i^n\}$  and  $D^n = \prod_{i \in N} D_i^n$  for every integer  $n$ , and let us prove that it is a weak strategic approximation of the initial game. We only have to prove that if  $(m, v) = \lim_{n \rightarrow +\infty} (m^n, u(m^n))$ , where  $(m^n, u(m^n))$  is a Nash equilibrium of the game restricted to  $D^n$  (for every integer  $n \geq 0$ ), then  $(m, v)$  is a Reny solution (which implies that it is an approximate equilibrium, since the game is approximately better-reply secure). By contradiction, assume  $(m, v)$  is not a Reny solution. By definition of  $V$  and  $V^n$  ( $n \geq 0$ ), there exists some integer  $k$  such that  $(m, v) \in V^k$  and  $(m^n, u(m^n)) \in V^k$  for every  $n$  large enough. By definition of  $V^k$ , there is some player  $i$  such that  $u_i(d_i^k, m'_{-i}) > v'_i$  for every  $(m', v') \in V^k$ . In particular,  $u_i(d_i^k, m^n_{-i}) > u_i(m^n)$  for  $n$  large enough, which contradicts the fact that  $(m^n, u(m^n))$  is a mixed Nash equilibrium of the game  $G'$  restricted to  $D^n$ , since  $d_i^k \in D_i^n$  for every  $n \geq k$ .

### A.5 Proof of Theorem 4.8

The proof of Theorems 4.8 and 4.12 uses the same principle: for every  $\varepsilon > 0$ , we construct a finite approximation  $G'_\varepsilon$  of  $G'$  such that for every player  $i$  and every mixed strategy of players  $j \neq i$  in  $G'_\varepsilon$ , player  $i$  has an  $\varepsilon$ -best response that belongs to  $G'_\varepsilon$ . This proves that every mixed Nash equilibrium of the finite approximation is an  $\varepsilon$ -Nash equilibrium of the initial game. The approximation shall depend on the structure of the game and, in particular, on the behavior of the payoffs in a neighborhood of the boundary.

First note that in two-player diagonal games, necessarily  $\phi(y) = y$  for every  $y \in [0, 1]$  (by the unanimity condition). Call  $x \in [0, 1]$  a right local equilibrium if  $h_i(x, x) > g_i(x, x)$  for both  $i = 1, 2$  and a left local equilibrium if  $h_i(x, x) > f_i(x, x)$  for both  $i = 1, 2$ . Thus, if players are supposed to play  $(x, x)$  and if  $x$  is a right local equilibrium, no player has an interest to deviate to some strategy in some right neighborhood of  $x$  (but he may have a profitable deviation outside that neighborhood) and similarly for left local equilibria.

We let  $x_0$  be the largest element in  $[0, 1]$  such that all  $x < x_0$  are right local equilibria and let  $y_0$  be the smallest element in  $[0, 1]$  such that all  $y > y_0$  are left local equilibria. Note that  $x_0$  may be 0 and  $y_0$  could be 1. By continuity of  $f$ ,  $g$ , and  $h$ , if  $x_0 < 1$ , then  $h_i(x_0, x_0) \leq g_i(x_0, x_0)$  for some  $i \in \{1, 2\}$ , and similarly, if  $y_0 > 0$ , then  $h_j(y_0, y_0) \leq f_j(y_0, y_0)$  for some  $j \in \{1, 2\}$ . Depending on the relative position of  $x_0$  and  $y_0$ , we consider the three following cases.

*First case:  $x_0 > y_0$ .* In this case, the finite approximated game is simply defined by some finite discretization  $D$  of  $[0, 1]$  containing 0 and 1 and  $\sigma$  a mixed strategy of the game restricted to  $D$ . Without any loss of generality, taking the mesh of this discretization smaller than some  $\eta > 0$ , we can assume that the payoff functions  $f$  and  $g$  do not change by more than  $\varepsilon/2$  if a player moves by no more than  $\eta$ , and such that if  $x < x_0$  is in  $D$ , then  $h_i(x, x) > g_i(y, x)$  for all  $x < y < x + \eta$ , and if  $x > y_0$  is in  $D$ , then  $h_i(x, x) > f_i(y, x)$  for all  $x > y > x - \eta$ . We let  $y \in [0, 1]$  be some  $\varepsilon/2$ -best reply to  $\sigma_j$  of player  $i$  that is not in  $D$  (if such strategy does not exist, we are done with the proof). Then either  $y < x_0$  or  $y > y_0$ . In the first case, denote by  $z$  the greatest element in  $D$  smaller than  $y$ , so that  $h_i(z, z) > g_i(y, z)$  by assumption of the discretization and since  $z$  is a right equilibrium. Since player  $j$  plays a probability distribution supported on  $D$ , moving from  $y$  to  $z$  for player  $i$  induces for him a greater payoff from the event associated to player  $j$  playing  $z$



and at most a change of  $\varepsilon/2$  on the events where player  $j$  is playing a strategy in  $D$  different from  $z$ . Thus,  $z$  is an  $\varepsilon$ -best reply for player  $i$ . A similar argument applies to  $y > y_0$  (use the left equilibrium property).

*Second case:*  $x_0 < y_0$ . This case implies that  $h_k(x_0, x_0) \leq g_k(x_0, x_0)$  and  $h_l(y_0, y_0) \leq f_l(y_0, y_0)$  for some  $k \in \{1, 2\}$  and  $l \in \{1, 2\}$ . By continuity, we get  $\eta > 0$  small enough such that  $h_k(x_0, x_0) < g_k(x, x_0) + \varepsilon/4$  for every  $x \in ]x_0, x_0 + \eta[$  and  $h_l(y_0, y_0) < f_l(y, y_0) + \varepsilon/4$  for every  $y \in ]y_0 - \eta, y_0[$ . Thus, there are four cases to check, depending on the values of  $k$  and  $l$ . Let us solve explicitly the case  $k = 1$  and  $l = 2$ . The same idea of construction could be done in the other cases, with a small adaptation in the definition of the weak strategic approximation.

Fix  $\varepsilon > 0$  and define  $x_0 = t_0 < s_0 < t_1 < \dots < s_{K-1} < t_K = y_0$ , a discretization of  $[x_0, y_0]$  with a mesh smaller than some  $\eta > 0$  so that payoff functions  $f$  and  $g$  do not change by more than  $\varepsilon/4$  if the pure strategy moves by no more than  $\eta$ . As in the first case, we let  $D$  be a finite discretization of  $[0, x_0[ \cup ]y_0, 1]$  with a mesh smaller than  $\eta > 0$  so that payoff functions  $f$  and  $g$  do not change by more than  $\varepsilon/2$  if the pure strategy moves by no more than  $\eta$  and such that if  $x < x_0$  is in  $D$ , then  $h_i(x, x) > g_i(y, x)$  for all  $x < y < x + \eta$ , and if  $x > y_0$  is in  $D$ , then  $h_i(x, x) > f_i(y, x)$  for all  $x > y > x + \eta$ . Now the finite approximation of  $G'$  is defined as follows: player 1 is restricted to play in  $D$  or uniformly on one of the intervals  $[t_k, s_k]$ ,  $k = 0, \dots, K - 1$ , or to choose  $t_K = y_0$ . Player 2 is restricted to play in  $D$  or uniformly on one of the intervals  $[s_k, t_{k+1}]$ ,  $k = 0, \dots, K - 1$ , or to choose  $t_0 = x_0$ . Observe that the intervals where players are uniformly mixing are disjoint and alternate. We let  $\sigma$  be some strategy of player 2 in the restricted game. Let  $y$  be some  $\varepsilon/4$  pure best response of player 1 in  $G$ , which is not in the discretization  $D$  (again, if it does not exist, this is finished). Several subcases have to be examined. *First subcase:* If  $y < x_0$  or  $y > y_0$ , we proceed as in the first case to construct an  $\varepsilon$ -best reply in  $D$ . *Second subcase:* If  $y$  is in some interval  $]s_k, t_{k+1}[$  of player 2 ( $k \in \{0, 1, \dots, K - 1\}$ ), and if player 2 is choosing that interval with positive probability, an easy computation proves that the payoff of player 1 coming from that interval is, up to  $\varepsilon/4$ , a convex combination of his payoff if he chooses  $t_{k+1}$  and his payoff if he chooses  $s_k$ . But the payoff of player 1 coming from player 2 playing in the other intervals or in  $D$  changes by no more than  $\varepsilon/4$  when he moves in the interval  $[s_k, t_{k+1}]$ . Consequently, player 1 has a  $\frac{3}{4}\varepsilon$ -best response at the extreme points  $s_k$  or  $t_{k+1}$  of the interval, a case that is treated in the next subcase. *Third subcase:* Let  $z \in [t_k, s_k]$  be a  $\frac{3}{4}\varepsilon$ -best response, for some  $k \in \{0, 1, \dots, K - 1\}$ . If  $k > 0$ , by assumption, there is zero probability that player 2 stops in that interval and so player 1's payoff does not move by more than  $\varepsilon/4$  if he plays uniformly in  $[t_k, s_k]$  (which is authorized for player 1) instead of playing  $z$ . This gives an  $\varepsilon$ -best response. If  $k = 0$ , if player 2 is playing  $x_0$  with positive probability and player 1 is playing  $z = x_0$ , then player 1 does not lose more than  $\varepsilon/4$  by playing slightly more than  $x_0$  instead of  $x_0$  (since  $h_1(x_0, x_0) < g_1(x, x_0) + \varepsilon/4$  for every  $x \in ]x_0, x_0 + \eta[$ ). Then the case where  $z$  belongs to the interval  $]t_0, s_0[$  remains. But, again, since his payoff moves continuously in that interval, playing uniformly in it is an  $\varepsilon$ -best response. The proof for player 2 is similar (we use the fact that  $h_2(y_0, y_0) < f_2(y, y_0) + \varepsilon/4$  for every  $y \in ]y_0 - \eta, y_0[$ ).

The three remaining cases for  $k$  and  $l$  are solved similarly, by a judicious choice of which of the two players is allowed to stop at  $x_0$  and  $y_0$ : if  $k = 2$  and  $l = 1$ , then player 1

can stop at  $x_0$  and player 2 at  $y_0$ ; if  $k = 2$  and  $l = 2$ , (only) player 1 is allowed to stop at both  $x_0$  and  $y_0$ ; if  $k = 1$  and  $l = 1$ , only player 2 is allowed to stop at both points. If some player can stop at  $x_0$ , then it is the other player who is authorized to stop uniformly in the small interval of the discretization just after  $x_0$ , and the intervals in which players can stop (by mixing uniformly) alternate until the point  $y_0$ , and the last interval belongs to the player who is not allowed to stop at  $y_0$ .

*Third case:*  $x_0 = y_0$ . This case implies  $h_k(x_0, x_0) < g_k(x, x_0) + \varepsilon/4$  for  $x \in ]x_0, x_0 + \eta[$  and  $h_l(x_0, x_0) < f_l(x, x_0) + \varepsilon/4$  for  $x \in ]x_0 - \eta, x_0[$  for some  $k \in \{1, 2\}$  and  $l \in \{1, 2\}$  (if  $x_0$  is 0 or 1, then only one of the inequalities holds). Suppose, for example, that  $h_1(x_0, x_0) < g_1(x, x_0) + \varepsilon/4$  for  $x \in ]x_0, x_0 + \eta[$ . We let  $D_1 = \{0 = t_0 < \dots < t_K\}$  be a discretization on the left of  $x_0$ , not including  $x_0$ , and empty if  $x_0 = 0$ ; let  $D_2 = \{s_0 < \dots < s_K = 1\}$  be a discretization on the right of  $y_0$ , not including  $y_0$ , and empty if  $y_0 = 1$ . Again, without any loss of generality, assume that the mesh of the discretizations is smaller than  $\eta > 0$ , so that payoff functions  $f$  and  $g$  do not change by more than  $\varepsilon/2$  if a player moves by no more than  $\eta$ . Consider a strategic approximation where player 2 is allowed to play in  $D_1 \cup D_2 \cup \{x_0\}$  and player 1 to play in  $D_1 \cup D_2$  or to mix uniformly in the length  $[x_0, s_0]$ . Let  $y \in [0, 1]$  be some  $\varepsilon/2$ -best reply of player 1 to some mixed strategy of player 2 that is not in  $D_1$  (if such a strategy does not exist, this is finished). If  $y < x_0$ , moving from  $y$  to the greatest element in  $D_1$  smaller than  $y$  gives an  $\varepsilon$ -best reply for player 1. If  $y > x_0$ , moving from  $y$  to the smallest element in  $D_1$  larger than  $y$  gives an  $\varepsilon$ -best reply for player 1. Last, if  $y = x_0$ , playing uniformly in  $[x_0, s_0]$  instead of playing  $x_0$  is an  $\varepsilon$ -best reply for player 1, because  $h_1(x_0, x_0) < g_1(x, x_0) + \varepsilon/4$  for  $x \in ]x_0, x_0 + \eta[$ . We treat in a similar way the case of player 2 and the case  $k = 2$ .

#### A.6 Proof of Theorem 4.12

Recall that for simplicity, we assume the type space  $T$  to be finite.

*Case (c2): the multiplayer private value setting.* Define a weak strategic approximation of the initial game  $G$  as follows: for each integer  $m$ , a strategy (in  $M_i$ ) of player  $i$  (whatever his type) is some element of the finite set  $D^m$  of uniform distributions on  $I_m^k = [k/m, (k+1)/m]$  ( $k \in \{0, 1, \dots, m-1\}$ ). By the Nash theorem, this finite (Bayesian) game has a mixed Nash equilibrium  $\sigma^m$ . We shall prove that if players  $j \neq i$  play according to  $\sigma_{-i}^m$ , each type  $t_i$  of player  $i$  has some  $\varepsilon$ -best response (in  $G'$ ) that belongs to  $D^m$ . This proves that  $\sigma^m$  is an  $\varepsilon$ -Nash equilibrium of  $G'$  for  $m$  large enough.

Consider  $\varepsilon > 0$ , and suppose  $m$  is large enough so that for every  $t \in T$ ,  $f_i(t, \cdot, y)$  and  $g_i(t, \cdot, y)$  do not move by more than  $\varepsilon$  in the interval  $[k/m, (k+2)/m]$  ( $k = 0, \dots, m-2$ ) uniformly in  $y$ . If player  $i$  of type  $t_i$  chooses a pure strategy  $x \in [0, 1]$  and if the *realized* strategy profile of its opponents is  $x_{-i}$ , then its payoff can be written  $w_i(t_i, x, \phi_i(x_{-i}))$ , where  $w_i(t_i, x, \phi_i(x_{-i}))$  is almost surely equal to either  $f_i(t_i, x, \phi_i(x_{-i}))$  or  $g_i(t_i, x, \phi_i(x_{-i}))$  (ties have zero probability), depending on the position of  $\phi_i(x_{-i})$  with respect to  $x$ . This is because the image probability measure of  $\sigma_{-i}^m$  by  $\phi_i$  has no atoms.<sup>17</sup> It also implies that the (expected) payoff of player  $i$ , fixing the strategies of the

<sup>17</sup>To prove this, consider the event  $[\phi_N(x_1, \dots, x_{N-1}) = \alpha]$  for some  $\alpha \in [0, 1]$ . Let  $S = \{(x_1, \dots, x_{N-1}) \in [0, 1]^{N-1} : \sum_{k=1}^{N-1} x_k = 1\}$  be the  $(N-2)$  simplex. The monotonicity assumption guarantees that for every

opponents, is a continuous function of his own strategy  $x$ . Consequently, there exists  $x^* \in [0, 1]$ , a pure best response of player  $t_i$  (in the original game  $G'$ ). From  $x^*$ , one can construct an  $\varepsilon$ -best response in  $D^m$  as follows: if  $x^* \in [0, 1/m]$ , from assumption (a), replacing  $x^*$  by the uniform distribution on  $I_m^1$  is an  $\varepsilon$ -best response for  $m$  large enough. If  $k/m < x^* < (k+1)/m$  for some  $k = 1, \dots, m-2$ , then either  $g_i(t_i, x^*, x^*) \geq f_i(t_i, x^*, x^*)$  and then replace  $x^*$  with the uniform strategy on  $I_m^{k+1}$  (it is better for  $i$  to choose a higher strategy), or  $g_i(t_i, x^*, x^*) < f_i(t_i, x^*, x^*)$  and then replace  $x^*$  with the uniform distribution on  $I_m^{k-1}$  (it is better for  $i$  to choose a lower strategy). In both cases, this gives an  $\varepsilon$ -best response in  $D^m$  for  $m$  large enough. Last, if assumption (b1) is satisfied and (b2) is not and  $x^* \in [1 - 1/m, 1]$ , then replace  $x^*$  with the uniform distribution on  $I_m^{m-1}$ .

REMARK A.1. Note that this proof works also when the payoff of type  $t_i$  depends also on  $(t_i, t_{-i})$  if we add the assumption that for every player  $i$  and every  $x^* \in [0, 1]$ , if  $g_i(t, x^*, x^*) \leq f_i(t, x^*, x^*)$  is true for one  $\bar{t}_{-i}$ , then it is true for every  $t_{-i}$ , and similarly for the inequality  $g_i(t, x^*, x^*) \geq f_i(t, x^*, x^*)$ . Finally, remark that the proof only requires the (strict) monotonicity of  $\phi_i$ , not the other properties.

*Case (c1): the two-player general value setting.* When there are two players, by unanimity  $\phi_i(y) = y$ . Now we mimic the proof and the approximation scheme of the second case of Theorem 4.8 with  $x_0 = 0$  and  $y_0 = 1$ , proving that if  $\sigma$  is some mixed strategy profile of player 2 in the approximated game, then every type  $t_1$  has an  $\varepsilon$ -best response against  $\sigma$  in the full game that belongs to his set of authorized strategies. That is, take the following discretization of  $[0, 1]$ :  $0 = s_0 < t_0 < s_1 < t_1 < \dots < t_K < s_{K+1} = 1$ . Player 1 is restricted to play uniformly on one of the intervals  $[s_k, t_k]$ ,  $k = 0, \dots, K$ , or to choose  $x = 1$ . Player 2 is restricted to play uniformly on one of the intervals  $[t_k, s_{k+1}]$ ,  $k = 0, \dots, K$ , or to choose  $x = 0$ . Observe that the intervals where players are mixing are disjoint and alternate (player 1 can stop uniformly in the first interval, player 2 in the second, player 1 in the third, etc.).

REMARK A.2. In both cases (two players or private value with  $N$  players), by construction, in the weak strategic approximation, ties have zero probability. Consequently, the Nash equilibria are independent on the tie-breaking rule  $h$ . Also, by examining the proofs, one can see that the result of Theorem 4.12 holds when the type space  $T$  is a compact metric set, functions  $f_i(t, a, b)$ ,  $g_i(t, a, b)$  are jointly continuous in  $(t, a, b) \in T \times [0, 1] \times [0, 1]$ , and  $h_i(t, x)$  is measurable and continuous in  $t$  uniformly in  $x$ . In that case, for every  $\varepsilon > 0$ , one can discretize the type space to obtain a Bayesian diagonal game with finitely many types  $t_\varepsilon \in T_\varepsilon$ . This game admits an  $\varepsilon$  equilibrium (by Theorem 4.12) from which one can construct a  $2\varepsilon$  equilibrium in the

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$(x_1, \dots, x_{N-1}) \in S$ , there is no more than one  $y \in \mathbb{R}$  such that  $\phi_N((x_1, \dots, x_{N-1}) + y(1, \dots, 1)) = \alpha$ . For every  $x = (x_1, \dots, x_{N-1}) \in S$ , define  $\psi(x_1, \dots, x_{N-1}) = y$  if such  $y$  exists and  $\psi(x_1, \dots, x_{N-1}) = 0$  otherwise. One can identify  $S \times \mathbb{R}$  to a subset of  $\mathbb{R}^{N-1}$  (which contains  $[0, 1]^{N-1}$ ) throughout the mapping  $i : S \times \mathbb{R} \rightarrow \mathbb{R}^{N-1}$  defined by  $i((x_1, \dots, x_{N-1}), y) = (x_1, \dots, x_{N-1}) + y(1, \dots, 1)$ . With this identification, the graph of  $\psi$  contains the event  $[\phi_N(x_1, \dots, x_{N-1}) = \alpha]$ . But  $\psi$  is measurable, and thus its graph has a 0 Lebesgue measure in  $\mathbb{R}^{N-1}$  (from the Fubini theorem). The assertion follows from the fact that  $\sigma_{-i}^m$  is a convex combination of uniform probabilities whose supports are rectangles with nonempty interiors.

original game by asking each type  $t_i$  in the original game to play a strategy of a closest type  $t_i^\varepsilon$  in the game with finitely many types. If the discretization is well chosen so that  $\sup_x |u_i(t_i, x) - u_i(t_i^\varepsilon, x)| \leq \varepsilon$ , then type  $t_i$  is playing a  $2\varepsilon$ -best response in the original game.

#### A.7 Proof of Example 4.16

Start with the maxmin. We let  $\alpha$  be the probability with which player 1 stops at  $x = 0$  (so with probability  $(1 - \alpha)$  he stops after zero). If  $\alpha = 0$ , player 2 gets 1 by stopping at time zero (and so player 1 gets  $-1$ ). If  $\alpha > 0$ , type  $A$  for player 2 can stop uniformly between 0 and some  $\varepsilon$  where  $\varepsilon$  is very small so that with high probability, if the game has not been stopped at time zero, it is stopped by player 2 (just after zero). Assume that type  $B$  of player 2 stops at time zero. Payoff of player 1 is thus very close to  $\alpha(\frac{1}{2} \times 1 + \frac{1}{2} \times -2) + (1 - \alpha) \times -1$ . Consequently, the best strategy for player 1 against such behavior by player 2 is to stop at  $t = 0$  with probability 1 so that  $\max \min \leq -\frac{1}{2}$ .

Let us now compute the min max. Let us restrict player 1 to playing best replies to the following kinds of strategies: (i) to stop at time  $t = 0$  or (2) to stop uniformly between 0 and some  $\varepsilon$  very small, which depends of course on the strategy of player 2. Knowing this behavior, type  $B$  must stop at time zero. We let  $\beta$  be the probability that type  $A$  stops at time zero. The payoff of player 1 if he stops at 0 (choose option 1) is  $\frac{1}{2} \times -2 + \frac{1}{2} \times (\beta \times 3 + (1 - \beta) \times 1) = \frac{-1}{2} + \beta$ , while if he chooses option (ii) his payoff is close to  $\frac{1}{2} \times -1 + \frac{1}{2}(\beta \times -1 + (1 - \beta) \times 1) = -\beta$ . Thus, the optimal  $\beta$  for type  $B$  against this behavior of player 1 must be equalizing and so is  $\beta = \frac{1}{4}$ . Consequently,  $\min \max \geq -\frac{1}{4}$ .

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