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# A unifying approach to incentive compatibility in moral hazard problems

René Kirkegaard
Department of Economics and Finance, University of Guelph

A new approach to moral hazard is presented. Once local incentive compatibility is satisfied, the problem of verifying global incentive compatibility is shown to be isomorphic to the problem of comparing two classes of distribution functions. Thus, tools from choice under uncertainty can be brought to bear on the problem. The approach allows classic justifications of the first-order approach (FOA) to be proven using the same unifying methodology. However, the approach is especially useful for analyzing higher-dimensional moral hazard problems. New and more tractable multi-signal justifications of the FOA are derived and implications for optimal monitoring are examined. The approach yields justifications of the FOA in certain settings where the action is multidimensional, as in the case when the agent is multitasking. Finally, a tractable multitasking model with richer predictions than the popular but simple linear-exponential-normal model is presented.

Keywords. First-order approach, moral hazard, multi-tasking, orthant orders, principal–agent models, stochastic orders.

JEL classification. D82, D86.

## 1. Introduction

The principal–agent model of moral hazard is among the core models of microeconomic theory and central to the economics of information. The problem is conceptually simple: a principal must design a contract to induce the agent to take the desired action. The intended action must be the agent's preferred action. Thus, a multitude of incentive compatibility constraints must be satisfied. Unfortunately, it is generally difficult to determine which constraints bind and to make robust predictions about the structure of optimal contracts. Moreover, these technical complications limit the scope of economic analysis. For instance, most of the literature assumes that the agent is responsible for one task only.

I propose an accessible approach to the moral hazard problem. Specifically, reformulating the problem makes it possible to utilize results from choice under uncertainty to shed light on the moral hazard problem. Many principal–agent results

René Kirkegaard: rkirkega@uoguelph.ca
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assume that the principal has access to a one-dimensional signal about the agent's one-dimensional action. In contrast, the new approach is particularly well suited to study higher-dimensional moral hazard problems. Thus, multiple signals can be more easily handled and it is also possible to tackle certain types of multitasking.

Most of the literature has focused on environments where the only binding constraint is the "local" incentive compatibility constraint (L-IC). In such cases, ensuring the agent has no incentive to deviate marginally from the intended action guarantees global incentive compatibility (G-IC), i.e., larger deviations are also unprofitable. Indeed, the classic first-order approach (FOA) uses the agent's first-order condition to summarize G-IC. The optimal contract is then easily derived. The FOA has a long history, dating back to Holmström (1979) and Mirrlees (1976, 1999). Rogerson (1985) and Jewitt (1988) have provided sufficient conditions under which the FOA is valid. However, although there are similarities in the structure of their proofs, the techniques they use are quite different. Similarly, Conlon (2009a) uses two different approaches to generalize Rogerson's and Jewitt's conditions to multi-signal environments.[1] All three papers assume there is a single task.

The starting point for the method proposed here is also L-IC. First, L-IC on its own imposes some structure on the contract. That structure is utilized to show that checking G-IC (once L-IC is satisfied) is isomorphic to the problem of comparing two risky prospects or two distribution functions. This is useful not only because choice under uncertainty is familiar, but also because it allows me to draw on a large literature in economics and statistics. Thus, once the isomorphisms have been established, most of the formal results in the paper follow by invoking well known results from the literature on stochastic orders. In this manner the method provides a unified methodology to understand Rogerson's, Jewitt's, and Conlon's classic results.

The link to choice under uncertainty can be understood by reexamining the differences between Rogerson's and Jewitt's contributions. In a one-signal–one-task setting, let $F(x|a)$ denote the distribution over the verifiable outcome ($x$) given the agent's action ($a$). Rogerson imposes the so-called convexity of the distribution function condition (CDFC), requiring that $F(x|a)$ is convex in $a$. In a first step, Rogerson establishes that as long as wages are monotonic in the outcome, the CDFC ensures that L-IC implies G-IC. The second step consists of identifying conditions that guarantee that the FOA wage schedule is in fact monotonic. Jewitt's proof consists of two similar steps, except he relies on contracts that give rise to utility that is increasing and concave in the outcome. This restriction allows him to weaken the CDFC; he assumes only that the antiderivative of $F(x|a)$ is convex in $a$. The top row in Table 1 contrasts these conditions. Note that the trade-off between the competing justifications is similar to the trade-off one faces when moving from first-order stochastic dominance (FOSD) to second-order stochastic dominance (SOSD) in the comparison of two risky prospects. See the second row of Table 1. There, a stronger assumption on the curvature of the utility function is also

[1]An earlier paper by Sinclair-Desgagné (1994) also extended Rogerson's conditions to the multi-signal model. Subsequently, Conlon (2009a) relaxed Sinclair-Desgagné's assumptions. Jewitt (1988) also offered two different multisignal justifications of the FOA. His results assume that there are exactly two signals and that they are independent. Conlon further generalized one of Jewitt's results.

| Rogerson | Jewitt |
|---|---|
| $F_{aa}(x\|a) \geq 0, \forall x, a$<br>$\Downarrow$<br>Any nondecreasing and L-IC contract is G-IC | $\int_{\underline{x}}^{x} F_{aa}(y\|a)\, dy \geq 0, \forall x, a$<br>$\Downarrow$<br>Any nondecreasing, concave, and L-IC contract is G-IC |

| FOSD | SOSD |
|---|---|
| $G(x) \leq H(x), \forall x$<br>$\Updownarrow$<br>$E_G[u(x)] \geq E_H[u(x)]$<br>for any nondecreasing $u(x)$ | $\int_{\underline{x}}^{x} G(y)\, dy \leq \int_{\underline{x}}^{x} H(y)\, dy, \forall x$<br>$\Updownarrow$<br>$E_G[u(x)] \geq E_H[u(x)]$<br>for any nondecreasing and concave $u(x)$ |

Table 1. Rogerson, Jewitt, and stochastic dominance. *Note*: The term $F(\cdot|a)$ is the distribution over outcomes given action $a$; the terms $G(x)$ and $H(x)$ are distribution functions.

exchanged for a weaker assumption on the antiderivative of the distribution functions. The method proposed in the current paper cements the analogy and thereby makes the point that stochastic orders are useful for the incentive compatibility problem.

The remainder of this introduction emphasizes the added economic insights that can be obtained by pursuing further justifications of the FOA.[2] Although it is possible to derive new justifications in the one-signal case (see Kirkegaard 2013), in this paper I concentrate on the more significant and economically meaningful extensions that are possible in multi-signal environments, with one or more tasks.

Even in the single-task case, Conlon's conditions become more restrictive as the number of signals increases. As explained below, this is due to the direction in which extensions into higher dimensions are pursued. However, there are several ways in which FOSD and SOSD can be extended from one dimension to many dimensions. Thus, new justifications can be obtained by exploiting other higher-dimensional extensions of FOSD and SOSD. Central to the new results are the so-called orthant orders. There are a number of advantages to these new justifications. Technically, they are in some ways more tractable. In economics terms, it is significant that they possess a certain additive property.[3]

To explain the additive property, assume the FOA is valid if the principal has access to one of two disjoint vectors of signals. I provide conditions under which the FOA remains valid when he gains access to both. The FOA can now be justified in the benchmark model where all signals are independent. Given independence, the conditions justifying

[2]The method has broader applicability. In Kirkegaard (2014), I utilize it to characterize the optimal contract in a specialized environment where the FOA is generally invalid.

[3]The new multivariate justifications are complements to Conlon's justifications since they are all derived from different generalizations of FOSD or SOSD. The new justifications impose weaker conditions on the distribution function. However, different assumptions on the likelihood ratio and the utility function must be made in each case.

the FOA do not become more restrictive as more signals are acquired. In contrast, Conlon notes that his so-called concave increasing-set probability (CISP) condition must fail if there are sufficiently many independent and identically distributed (i.i.d.) signals. Thus, this environment represents a concrete example in which the approach is useful beyond existing results.

Along with the other multi-signal results, the additive property provides a partial answer to a question left open since Holmström (1979). Holmström (1979) studies when additional signals are valuable and how they alter the optimal contract. However, he simply assumes that the FOA is and remains valid; the additive property illustrates when this assumption is legitimate. Thus, the additive property is relevant for the principal's incentive to invest in more sources of information. The new justifications of the FOA invite a reexamination of such endogenous monitoring. While Holmström (1979) does not analyze how the induced (second-best) action changes, I show that it may be nonmonotonic in the number of signals. Thus, the optimal action may move further away from the first best as another signal is acquired. The reason is that although additional signals make it cheaper to induce any given action, these cost savings are not constant across actions. If the additional savings are sufficiently asymmetric across actions, a new signal may then further distort the second best away from the first best.

The final contribution of the paper is to relax the assumption that the agent performs a single task. I concentrate on a specialized model in which different tasks produce independent vectors of signals. In this case, the tasks are interdependent only through the agent's cost function. A counterpart to the additive property holds: the FOA remains valid under familiar conditions as more and more tasks are added. Thus, the FOA can be used to analyze, e.g., how many tasks the agent should be assigned or how to distribute a given number of tasks across a set of agents. To the best of my knowledge, these are the first multi-task justifications of the FOA.

The commonly applied multi-task model is a static model often referred to as the linear-exponential-normal (LEN) model. Holmström and Milgrom (1987) provide a dynamic foundation for such a model, but at the same time they warn against "indiscriminate application" of it. An important implicit assumption is that the informational quality of the signals is completely independent of the agent's action. Here, I characterize another simple multi-task model, the square root, independent tasks (SQIT) model. The LEN and SQIT models are both highly tractable. The SQIT model, however, allows the quality of the information system to depend on the action. As a result, it can be demonstrated that several of the equilibrium properties of the LEN model are not robust. Due to space restrictions, the multi-task results are outlined in Section 7; details are provided in the Appendix, available in a supplementary file on the journal website, http://econtheory.org/supp/2008/supplement.pdf.

## 2. Model and preliminaries

A risk averse agent takes a costly action that is not verifiable to others. In the general setting, the action may be $m$-dimensional, $m \geq 1$. Let $\mathbf{a} = (a_1, \ldots, a_m)$ denote the action, and assume that $a_j$ belongs to a closed and bounded interval, $[\underline{a}_j, \overline{a}_j]$, $j = 1, \ldots, m$.

Define $\mathcal{A} = \times_{j=1}^{m}[\underline{a}_j, \overline{a}_j]$ and let $\text{int}(\mathcal{A})$ denote the interior of this set. The agent's action determines the joint distribution of $n \geq 1$ verifiable signals, denoted $\mathbf{x} = (x_1, \ldots, x_n)$. If the action is $\mathbf{a}$, the cumulative distribution function is $F(\mathbf{x}|\mathbf{a})$, where it is assumed that the domain, $\mathcal{X} = \times_{i=1}^{n}[\underline{x}_i, \overline{x}_i]$, is convex, compact, and independent of $\mathbf{a}$. Define $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)$. To rule out degenerate cases, assume that if $\mathbf{a} \neq \mathbf{a}'$, then $F(\mathbf{x}|\mathbf{a}) \neq F(\mathbf{x}|\mathbf{a}')$ on a subset of $\mathcal{X}$ of strictly positive measure. It is assumed that $F(\mathbf{x}|\mathbf{a})$ has no mass points and is continuously differentiable in $\mathbf{x}$ and $\mathbf{a}$ to the requisite degree (see below), with $f(\mathbf{x}|\mathbf{a})$ denoting the density for fixed $\mathbf{a}$. Assume that $f(\mathbf{x}|\mathbf{a})$ is strictly positive. Let $\overline{F}(\mathbf{x}|\mathbf{a})$ denote the survival function, i.e., the probability that the vector of signals is greater than $\mathbf{x}$. Generally, $\overline{F}(\mathbf{x}|\mathbf{a}) \neq 1 - F(\mathbf{x}|\mathbf{a})$ when $n > 1$. Reference to $i$ (or $j$) will be suppressed when $n = 1$ (or $m = 1$). Partial derivatives of functions of many variables will be denoted by subscripts. Derivatives of single-variable functions are denoted with primes, double primes, etc.

The agent faces a contract that, to him, is fixed. He receives a deterministic wage $w(\mathbf{x})$ if the outcome is $\mathbf{x}$, in which case utility is $v(w(\mathbf{x})) - c(\mathbf{a})$.[4] The utility function $v(w)$ is strictly increasing and differentiable $2n + 1$ times. The agent is strictly risk averse, or $v''(\cdot) < 0$. The domain of the utility function is some interval that may or may not be the entire real line. Finally, assume $c(\mathbf{a})$ is differentiable, strictly increasing, and (weakly) convex. The agent's expected utility (assuming it exists) given action $\mathbf{a}$ is

$$EU(\mathbf{a}) = \int v(w(\mathbf{x}))f(\mathbf{x}|\mathbf{a})\,d\mathbf{x} - c(\mathbf{a}).$$

If the principal wishes to induce action $\mathbf{a}^* \in \mathcal{A}$, this action must provide the agent with higher expected utility than any other action, or

$$EU(\mathbf{a}^*) \geq EU(\mathbf{a}) \quad \text{for all } \mathbf{a} \in \mathcal{A}, \tag{G-IC$_{\mathbf{a}^*}$}$$

in which case the contract $w(\mathbf{x})$ is said to be globally incentive compatible. If $\mathbf{a}^* \in \text{int}(\mathcal{A})$, a minimum requirement is that $EU(\mathbf{a})$ attains a stationary point at $\mathbf{a}^*$, or

$$\int v(w(\mathbf{x}))f_{a_j}(\mathbf{x}|\mathbf{a}^*)\,d\mathbf{x} - c_j(\mathbf{a}^*) = 0 \quad \text{for all } j = 1, \ldots, m. \tag{L-IC$_{\mathbf{a}^*}$}$$

The stationary point may in principle be a local minimum or a saddle point. Nevertheless, I refer to the above condition as the local incentive compatibility condition. Thus, any contract that satisfies (L-IC$_{\mathbf{a}^*}$) will be termed L-IC$_{\mathbf{a}^*}$ and any contract that satisfies (G-IC$_{\mathbf{a}^*}$) is G-IC$_{\mathbf{a}^*}$. For expositional purposes, in this paper I follow the literature and focus on actions in the interior of $\mathcal{A}$. The implementation of actions on the boundary is discussed briefly in Kirkegaard (2013).

[4] Additive separability is important. While it is a standard assumption in the literature, there are exceptions. For instance, Alvi (1997) and Fagart and Fluet (2013) provide conditions that justify the FOA without additive separability.

### 2.1 *The FOA contract*

As much of the paper revolves around the FOA, it is worthwhile to set the stage by characterizing the FOA contract. Following Jewitt (1988), assume the principal is risk neutral. Let $B(\mathbf{a})$ denote the expected gross benefit to the principal if the agent's action is $\mathbf{a}$. In many one-signal–one-task applications, $B(a)$ is the expected value of $x$. Apart from incentive compatibility, the only other constraint is a participation constraint. Let $\overline{u}$ denote the agent's reservation utility. It will be assumed that the constraint set is nonempty. That is, there exists a contract that satisfies both the participation constraint and L-IC for some $\mathbf{a}$.

The FOA relies on L-IC being sufficient for G-IC. If this is the case, the principal's problem can be written as

$$\max_{w,\mathbf{a}} B(\mathbf{a}) - \int w(\mathbf{x}) f(\mathbf{x}|\mathbf{a})\, d\mathbf{x}$$

$$\text{st.} \quad \int v(w(\mathbf{x})) f(\mathbf{x}|\mathbf{a})\, d\mathbf{x} - c(\mathbf{a}) \geq \overline{u}$$

$$\int v(w(\mathbf{x})) f_{a_j}(\mathbf{x}|\mathbf{a})\, d\mathbf{x} - c_j(\mathbf{a}) = 0 \quad \text{for all } j = 1, \dots, m.$$

To proceed, let $l(\mathbf{x}|\mathbf{a}) \equiv \ln f(\mathbf{x}|\mathbf{a})$ and assume that

$$l_{a_j}(\mathbf{x}|\mathbf{a}) = \frac{\partial \ln f(\mathbf{x}|\mathbf{a})}{\partial a_j} = \frac{f_{a_j}(\mathbf{x}|\mathbf{a})}{f(\mathbf{x}|\mathbf{a})}$$

is bounded below, $j = 1, \dots, m$. Assume also that

$$\frac{\partial^{k_1+\cdots+k_n} l_{a_j}(\mathbf{x}|\mathbf{a})}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

exists for all $k_i \in \{0, 1, 2\}$, $i = 1, \dots, n$, with $k_1 + \cdots + k_n \geq 1$. Finally, assume in the remainder of the paper that the optimal wage, $w(\mathbf{x})$, is in the interior of the domain of $v(\cdot)$. This assumption is typically satisfied if the agent's reservation utility is high enough.[5] In this case, $w(\mathbf{x})$ is characterized by a first-order condition that can be written

$$\frac{1}{v'(w(\mathbf{x}))} = \lambda + \sum_{j=1}^{m} \mu_j l_{a_j}(\mathbf{x}|\mathbf{a}^*), \tag{1}$$

where $\lambda$ is the multiplier of the participation constraint and $\mu_j$ are the multipliers of the $m$-dimensional local incentive compatibility constraint. Note that $\lambda > 0$, or the right hand side of (1) would be nonpositive for some realizations of $\mathbf{x}$.

While (1) provides an implicit characterization of the contract $w(\mathbf{x})$, the agent is ultimately concerned with the payoff $v(w(\mathbf{x}))$ he receives when the vector of signals is $\mathbf{x}$. To

[5] See, e.g., Jewitt et al. (2008), and in particular Gutiérrez (2012) for a detailed discussion. As can be seen from (1), below, this also explains why the likelihood ratio must be bounded below.

aid the analysis, Jewitt defines the function

$$\omega(z) = v((v')^{-1}(1/z)), \quad z > 0,$$ [6]

so that, from (1), the agent's utility can be more compactly written as

$$v(w(\mathbf{x})) = \omega\left(\lambda + \sum_{j=1}^{m} \mu_j l_{a_j}(\mathbf{x}|\mathbf{a}^*)\right). \tag{2}$$

One advantage of Jewitt's formulation is that it makes it evident that it is the properties of the function $\omega(\cdot)$—along with the properties of $\mu_j l_{a_j}(\mathbf{x}|\mathbf{a})$—that determine for example the slope and curvature of the (composite) function of interest, $v(w(\mathbf{x}))$. Note that $\omega'(z) > 0$ since $v''(w) < 0$. When $\mathbf{a}$ is one-dimensional, Jewitt proves that $\mu > 0$ in any L-IC contract that takes the form in (1). Jewitt's argument does not rely on the FOA being valid. However, knowing that $\mu > 0$ in any contract that is derived using the FOA makes it possible to infer important qualitative properties of the FOA candidate contract.

In particular, when $\mathbf{a}$ is one-dimensional and $\mu > 0$, $v(w(\mathbf{x}))$ is increasing in its arguments if $l_a(\mathbf{x}|a)$ is increasing in each signal. The latter property is referred to as the monotone likelihood ratio property (MLRP). Jewitt imposes the added assumptions that $\omega$ is concave and that $l_a(x|a)$ is concave in $x$. Then $v(w(x))$ is increasing and concave. Example 1 illustrates Jewitt's assumption on $\omega$.

Example 1. Consider the utility functions

$$v^\alpha(w) = 1 - e^{-\alpha w}, \qquad v^\beta(w) = \frac{1}{\beta} w^\beta, \qquad v^0(w) = \ln w,$$

where $\alpha > 0$ and $\beta < 1$, with $\beta \neq 0$. The domain of the first function is $(-\infty, \infty)$ while the domain of the latter two is $(0, \infty)$ (or convex subsets thereof). The first utility function exhibits constant absolute risk aversion, while the other two exhibit constant relative risk aversion. For these functions,

$$\omega^\alpha(z) = 1 - \frac{1}{\alpha z}, \qquad \omega^\beta(z) = \frac{1}{\beta}(z)^{\beta/(1-\beta)}, \qquad \omega^0(z) = \ln z,$$

respectively. Thus, the first and third functions satisfy Jewitt's assumption that $\omega$ is concave. The second function satisfies the assumption if and only if $\beta \leq 0.5$. ◊

## 3. From local to global incentive compatibility

This section develops an alternative approach to moral hazard. To outline the main idea, assume first that the action is one-dimensional. Imagine the principal wants to induce action $a^*$. G-IC$_{a^*}$ requires that $EU(a^*) \geq EU(a)$ for all $a$, thus necessitating a continuum of comparisons of utility across different actions. In contrast, L-IC$_{a^*}$ just imposes the necessary condition that $EU(a)$ has a stationary point at $a^*$, or $EU'(a^*) = 0$. Note that

[6]To clarify, $(v')^{-1}$ refers to the inverse of $v'(\cdot)$, not the reciprocal.

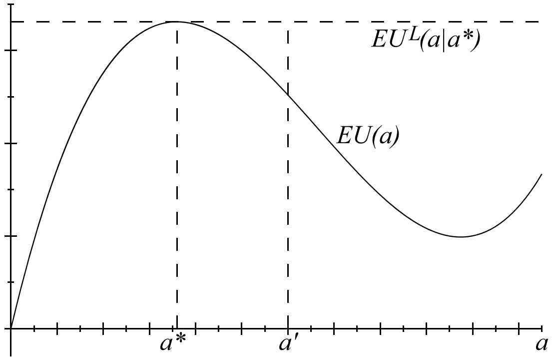


Figure 1. A new approach to moral hazard. Part I. *Note*: There are two equivalent ways to check that the agent will not deviate from $a^*$ to $a'$: (i) moving horizontally, $EU(a^*)$ must exceed $EU(a')$, and (ii) moving vertically, $EU^L(a'|a^*)$ must exceed $EU(a')$.

L-IC$_{a^*}$ implies that the tangent line to $EU(a)$ at $a^*$ is horizontal. Stated differently, the tangent line, $EU^L(a|a^*) = EU(a^*) + (a - a^*)EU'(a^*)$, takes a constant value. Thus, G-IC is obtained if and only if $EU(a)$ is always below this tangent line, or $EU(a) \leq EU^L(a|a^*)$ for all $a$. Figure 1 illustrates.

Based on this observation, I construct an auxiliary problem in which the agent's action $a$ manipulates imaginary distribution and cost functions, but which (given any L-IC$_{a^*}$ contract) yields constant expected utility that coincides with $EU(a^*) = EU^L(a|a^*)$. Checking G-IC$_{a^*}$ is then equivalent to checking that regardless of his action, the agent is better off in the auxiliary problem than in the real problem. Thus, rather than comparing expected utility across different actions, I compare expected utility for a given action across two different problems. By construction, costs will be lower in the auxiliary problem than in the real problem. Hence, the comparison between the two problems essentially reduces to comparing two risky prospects, described by the real and imaginary distribution functions, respectively. Thus, the incentive compatibility problem is turned into a mere exercise in choice under uncertainty. Finally, note that there are no conceptual difficulties in introducing multidimensional actions; the tangent line is simply replaced by the tangent plane.

### 3.1 *The auxiliary problem*

The function $EU^L(\mathbf{a}|\mathbf{a}^*)$ (see Figure 1) is obtained by linearizing $EU(\mathbf{a})$ around $a^*$. To develop a fitting auxiliary problem, it is therefore natural to start by linearizing the primitives of the problem, $F(\mathbf{x}|\mathbf{a})$ and $c(\mathbf{a})$.

Fix $\mathbf{a}^* \in \text{int}(\mathcal{A})$. Think of $\mathbf{a}^*$ as the action the principal seeks to implement.

Next, fix $\mathbf{x}$ as well and think of $\mathbf{a} = (a_1, \ldots, a_m)$ as the variables. Let

$$f^L(\mathbf{x}|\mathbf{a}, \mathbf{a}^*) = f(\mathbf{x}|\mathbf{a}^*) + \sum_{j=1}^{m} (a_j - a_j^*) f_{a_j}(\mathbf{x}|\mathbf{a}^*)$$

and

$$F^L(\mathbf{x}|\mathbf{a},\mathbf{a}^*) = F(\mathbf{x}|\mathbf{a}^*) + \sum_{j=1}^{m}(a_j - a_j^*)F_{a_j}(\mathbf{x}|\mathbf{a}^*)$$

be the tangent planes to $f(\mathbf{x}|\mathbf{a})$ and $F(\mathbf{x}|\mathbf{a})$, respectively, at $\mathbf{a} = \mathbf{a}^*$. Similarly, let

$$c^L(\mathbf{a}|\mathbf{a}^*) = c(\mathbf{a}^*) + \sum_{j=1}^{m}(a_j - a_j^*)c_j(\mathbf{a}^*).$$

Now switch the roles of $\mathbf{x}$ and $\mathbf{a}$. Holding $\mathbf{a}$ and $\mathbf{a}^*$ fixed, note that $F^L(\mathbf{x}|\mathbf{a},\mathbf{a}^*)$ need not be monotonic in $\mathbf{x}$, nor need it be bounded between 0 and 1. Nevertheless, the following thought experiment is proposed. Think of $f^L(\mathbf{x}|\mathbf{a},\mathbf{a}^*)$ and $F^L(\mathbf{x}|\mathbf{a},\mathbf{a}^*)$ as (admittedly odd) density and distribution functions, respectively. It is easy to see that $F^L$ can be obtained by integrating $f^L$ over $\mathbf{x}$. Likewise, $f^L$ integrates to 1, with $F^L(\underline{\mathbf{x}}|\mathbf{a},\mathbf{a}^*) = 0$ and $F^L(\overline{\mathbf{x}}|\mathbf{a},\mathbf{a}^*) = 1$. To see this, recall that $F(\underline{\mathbf{x}}|\mathbf{a}) = 0$ and $F(\overline{\mathbf{x}}|\mathbf{a}) = 1$ for all $\mathbf{a}$. Hence, $F_{a_j}(\underline{\mathbf{x}}|\mathbf{a}) = F_{a_j}(\overline{\mathbf{x}}|\mathbf{a}) = 0$. Now consider the imaginary problem where the agent faces distribution function $F^L(\mathbf{x}|\mathbf{a},\mathbf{a}^*)$ rather than $F(\mathbf{x}|\mathbf{a})$ and cost function $c^L(\mathbf{a}|\mathbf{a}^*)$ rather than $c(\mathbf{a})$.

Note that regardless of the contract, the imaginary problem exhibits constant returns to scale in the sense that expected utility is linear in $a_j$. Thus, any returns to scale in the original problem are filtered out. By contrasting the two problems, it may be possible to determine whether the original problem exhibits a form of decreasing returns to scale. Intuitively at least, such decreasing returns would suggest that the stationary point at $\mathbf{a}^*$ identifies the agent's optimal action. In fact, Rogerson remarks that "the CDFC is a form of stochastic diminishing returns to scale."

"Expected utility" in the auxiliary problem is

$$EU^L(\mathbf{a}|\mathbf{a}^*) = \int v(w(\mathbf{x}))f^L(\mathbf{x}|\mathbf{a},\mathbf{a}^*)\,d\mathbf{x} - c^L(\mathbf{a}|\mathbf{a}^*)$$

or

$$EU^L(\mathbf{a}|\mathbf{a}^*) = EU(\mathbf{a}^*) + \sum_{j=1}^{m}(a_j - a_j^*)\left[\int v(w(\mathbf{x}))f_{a_j}(\mathbf{x}|\mathbf{a}^*)\,d\mathbf{x} - c_j(\mathbf{a}^*)\right]. \tag{3}$$

Evidently, the last term disappears if L-IC$_{\mathbf{a}^*}$ is satisfied, in which case $EU^L(\mathbf{a}|\mathbf{a}^*) = EU(\mathbf{a}^*)$ for all $\mathbf{a}$. Stated differently, L-IC$_{\mathbf{a}^*}$ *on its own* places a lot of structure on the contract, and thus on $EU^L(\mathbf{a}|\mathbf{a}^*)$. In particular, it follows from (3) that once L-IC$_{\mathbf{a}^*}$ is satisfied, G-IC$_{\mathbf{a}^*}$ can equivalently be expressed as the requirement that

$$EU^L(\mathbf{a}|\mathbf{a}^*) \geq EU(\mathbf{a}) \quad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{4}$$

Hence, (4) captures the logic in Figure 1, as described at the beginning of this section. Note that (4) can be written as

$$\int v(w(\mathbf{x}))f^L(\mathbf{x}|\mathbf{a},\mathbf{a}^*)\,d\mathbf{x} - c^L(\mathbf{a}|\mathbf{a}^*) \geq \int v(w(\mathbf{x}))f(\mathbf{x}|\mathbf{a})\,d\mathbf{x} - c(\mathbf{a}) \quad \text{for all } \mathbf{a} \in \mathcal{A}.$$

The tangent plane to any convex function is below the function itself. Thus, since $c(\mathbf{a})$ is convex, it follows that $c^L(\mathbf{a}|\mathbf{a}^*) \leq c(\mathbf{a})$ for all $\mathbf{a} \in \mathcal{A}$. Hence, to obtain (4), it is sufficient that

$$\int v(w(\mathbf{x}))f^L(\mathbf{x}|a, a^*)\, d\mathbf{x} \geq \int v(w(\mathbf{x}))f(\mathbf{x}|a)\, d\mathbf{x} \quad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{5}$$

In essence, the continuum of incentive compatibility constraints in the original problem has been replaced with a continuum of comparisons of risky prospects. For instance, if $v(w(\mathbf{x}))$ is monotonic, it is fruitful to ask whether $F^L$ first-order stochastically dominates $F$. Since such comparisons are commonplace in economics, a large literature may now be accessed to inform the analysis. This reformulation of the problem constitutes the main methodological contribution of this paper. Proposition 1 formally records the conclusion.

Proposition 1. *Fix $\mathbf{a}^* \in \text{int}(\mathcal{A})$. Any L-IC$_{\mathbf{a}^*}$ contract is G-IC$_{\mathbf{a}^*}$ if (5) holds.*[7]

Note that (5) is satisfied if and only if

$$\int v(w(\mathbf{x}))[\kappa + \varepsilon f^L(\mathbf{x}|\mathbf{a}, \mathbf{a}^*)]\, d\mathbf{x} \geq \int v(w(\mathbf{x}))[\kappa + \varepsilon f(\mathbf{x}|\mathbf{a})]\, d\mathbf{x} \quad \text{for all } \mathbf{a} \in \mathcal{A} \tag{6}$$

for all $\varepsilon > 0$ and all $\kappa$. It is trivial to select $\kappa$ and $\varepsilon > 0$ in such a manner that both bracketed terms are proper densities, i.e., they are strictly positive and integrate to 1. The equivalence of (5) and (6) implies that even though $f^L$ is not a proper density, stochastic dominance results can still be invoked. Thus, I frequently abuse terminology and say that $f^L$ dominates $f$ in some stochastic order.

### 3.2 *Justifying the FOA with one signal and one task*

Consider the one-signal case with a one-dimensional action, or $n = m = 1$. If $F(x|a)$ is convex in $a$ for all $x$, then its tangent line, $F^L(x|a, a^*)$, lies everywhere below the function itself. Thus, $F^L(\cdot|a, a^*)$ first-order stochastically dominates $F(\cdot|a)$ for all $a$. Consequently, any L-IC$_{a^*}$ contract that is monotonic must be G-IC$_{a^*}$. Moreover, the argument holds regardless of $a^*$. Thus, if it can be established that the FOA candidate contract is monotonic, then the FOA is itself valid. Figure 2 illustrates.

The following lemma notes necessary and sufficient conditions for $F^L$ to first- or second-order stochastically dominate $F$ regardless of $(a, a^*)$.

Lemma 1 (Ordering the real and auxiliary distributions). *Assume $n = m = 1$. Then $F^L(\cdot|a, a^*)$ first-order stochastically dominates $F(\cdot|a)$ for all $a \in [\underline{a}, \overline{a}]$ and all $a^* \in [\underline{a}, \overline{a}]$ if and only if*

$$F_{aa}(x|a) \geq 0 \quad \textit{for all } x \in [\underline{x}, \overline{x}] \textit{ and all } a \in [\underline{a}, \overline{a}].$$

[7] If $c(\mathbf{a})$ is a linear function, then $c^L(\mathbf{a}|\mathbf{a}^*) = c(\mathbf{a})$ and so any L-IC$_{\mathbf{a}^*}$ contract is G-IC$_{\mathbf{a}^*}$ if and only if (5) holds. When the action is one-dimensional, or $m = 1$, it is common to normalize $c(a) = a$. See, e.g., Rogerson (1985) or Conlon (2009a). However, when the action is multidimensional, the cost function must generally be rich enough to capture interdependencies between the various dimensions.

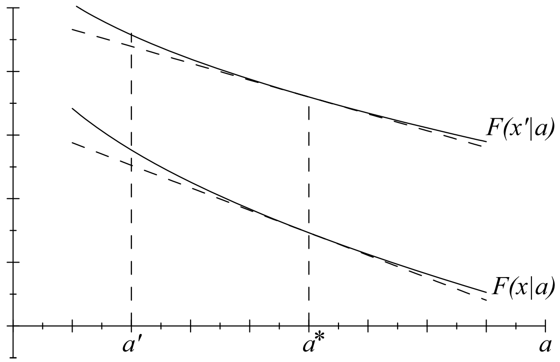


FIGURE 2. A new approach to moral hazard. Part II. *Note*: STEP 1. Fix $a^*$. For each $x$, construct the tangent line to $F(x|a)$ at $a^*$ (moving horizontally). STEP 2. For each $a \neq a^*$, like $a'$, move vertically to trace out the cumulative distribution function (cdf) in the auxiliary and real problems. Here, $F^L$ FOSD $F$ ($F^L$ lies always below $F$). Thus, any monotonic and L-IC$_{a^*}$ contract yields $EU(a^*) = EU^L(a^*|a^*) = EU^L(a'|a^*) \geq EU(a')$. STEP 3. To validate the FOA, the conclusion in Step 2 must hold regardless of $a^*$.

*Second, $F^L(\cdot|a, a^*)$ second-order stochastically dominates $F(\cdot|a)$ for all $a \in [\underline{a}, \overline{a}]$ and all $a^* \in [\underline{a}, \overline{a}]$ if and only if*

$$\int_{\underline{x}}^{x} F_{aa}(y|a)\, dy \geq 0 \quad \textit{for all } x \in [\underline{x}, \overline{x}] \textit{ and all } a \in [\underline{a}, \overline{a}].$$

PROOF. The first part follows from the fact that a function is convex if and only if it lies everywhere above all of its tangent line. For the second part, $\int_{\underline{x}}^{x} F(y|a)\, dy$ is likewise everywhere above its tangent line (as a function of $a$) if and only if it is convex, or $\int_{\underline{x}}^{x} F_{aa}(y|a)\, dy \geq 0$. Now, the tangent line to $\int_{\underline{x}}^{x} F(y|a)\, dy$ at $a = a^*$ is

$$\int_{\underline{x}}^{x} F(y|a^*)\, dy + (a - a^*) \int_{\underline{x}}^{x} F_a(y|a^*)\, dy = \int_{\underline{x}}^{x} F^L(y|a, a^*)\, dy.$$

It follows that $\int_{\underline{x}}^{x} F_{aa}(y|a)\, dy \geq 0$ for all $x \in [\underline{x}, \overline{x}]$, and all $a \in [\underline{a}, \overline{a}]$ is necessary and sufficient for $\int_{\underline{x}}^{x} F^L(y|a, a^*)\, dy \leq \int_{\underline{x}}^{x} F(y|a)\, dy$ for all $x$, $a$, and $a^*$. Of course, for fixed $(a, a^*)$, the latter condition coincides with the definition that $F^L(\cdot|a, a^*)$ second-order stochastically dominates $F(\cdot|a)$. □

Many results of the type presented in Lemma 1 are utilized in the analysis. Since the proofs are trivial and in any event analogous to the proof of Lemma 1, the formal proofs are omitted. Recall that Rogerson's (1985) CDFC assumption is that $F_{aa}(x|a) \geq 0$. The condition $\int_{\underline{x}}^{x} F_{aa}(y|a)\, dy \geq 0$ is Jewitt's (1988) assumption (2.10a).

To make use of FOSD it is necessary that the candidate contract is monotonic. To utilize SOSD, the contract must be monotonic and concave. The discussion preceding Example 1 identifies conditions under which these properties are obtained. The validity of the FOA follows by imposing both sets of assumptions. Recall that $\omega'(z) > 0$ follows automatically from risk aversion.

Proposition 2. *Assume $n = m = 1$ and that the second-best action is in $(\underline{a}, \overline{a})$. Then the FOA is valid if either of the following statements holds:*

*(i) We have $F_{aa}(x|a) \geq 0$ for all $x$ and $a$, $\omega'(\cdot) > 0$, and $l_{ax}(x|a) \geq 0$ for all $x$ and $a$.*

*(ii) We have $\int_{\underline{x}}^{x} F_{aa}(y|a)\,dy \geq 0$ for all $x$ and $a$, $\omega'(\cdot) > 0 \geq \omega''(\cdot)$, and $l_{ax}(x|a) \geq 0 \geq l_{axx}(x|a)$ for all $x$ and $a$.*

The conditions in Proposition 2 coincide with Rogerson's and Jewitt's conditions. However, it is not hard to generalize the proposition to utilize stochastic dominance of higher order (e.g., third-order stochastic dominance). See Kirkegaard (2013) for details. As with Jewitt's justification, these extensions necessitate that $\omega(\lambda + \mu l_a(x|a))$ is increasing and concave. Kirkegaard (2013) also derives justifications for the FOA in environments where $\omega$ is convex (note that $\omega$ may be convex even if $v$ is concave).

## 4. Multi-signal justifications of the FOA

The remainder of the paper extends Proposition 2 to higher dimensions. Section 4.1 offers some observations on how univariate stochastic orders may be generalized to obtain relevant multivariate stochastic orders. Pertinent properties of the FOA contract are identified and sufficient conditions for those properties to hold are derived. Sections 4.2 and 4.3 derive new justifications for the FOA when there are multiple signals but the action is one-dimensional, or $n > 1 = m$.

### 4.1 *Integral stochastic orders*

The stochastic orders invoked in this paper are *integral stochastic orders*. The distribution function $G$ dominates the distribution function $H$ if $G$ is weakly preferred to $H$ for all utility functions in some class $\mathcal{U}$, where $\mathcal{U}$ is referred to as a "generator" of the integral stochastic order. For an introduction to integral stochastic orders, see Müller and Stoyan (2002). In this section I introduce two classes of utility functions that are relevant for the moral hazard problem.

Definition 1 (Multivariate $\Delta$-antitone function). A multivariate function with $n$ variables, $u(\mathbf{x})$, is $\Delta$-*antitone* if

$$\frac{\partial^{k_1+\cdots+k_n}[-u(-\mathbf{x})]}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}} \geq 0 \tag{7}$$

for all $k_i \in \{0, 1\}$, $i = 1, \ldots, n$, with $k_1 + \cdots + k_n \geq 1$.[8]

Definition 2 (Multivariate $\Delta^2$-antitone function). A multivariate function with $n$ variables, $u(\mathbf{x})$, is $\Delta^2$-*antitone* if

$$\frac{\partial^{k_1+\cdots+k_n}[-u(-\mathbf{x})]}{\partial x_1^{k_1}\cdots\partial x_n^{k_n}} \geq 0$$

[8] This is with some abuse of terminology, as a function is more commonly said to be $\Delta$-antitone if the derivatives of $u(-\mathbf{x})$ (rather than $-u(-\mathbf{x})$) have the property in (7). Difference operators can be used to extend this and the following definition to nondifferentiable functions; see Müller and Stoyan (2002).

for all $k_i \in \{0, 1, 2\}$, $i = 1, \dots, n$, with $k_1 + \cdots + k_n \geq 1$.

Thus, a multivariate function is Δ-antitone if the cross-partial derivatives alternate in sign as more cross-partials are added. In the bivariate case, for example, the marginal utility of $x_1$ is smaller the higher $x_2$ is. The term $i$-antitone is used to refer to a univariate function for which the first $i$ derivatives of $-u(-x)$ are nonnegative. Note that this implies that the derivatives of $u(x)$ alternate in sign, i.e., $(-1)^{k-1}u^{(k)} \geq 0$ for all $k = 1, 2, \dots, i$, where $u^{(k)}$ denotes the $s$th derivative.

The composite utility function in (2) is of primary interest. Note that when $a$ is one-dimensional, Jewitt's proof that $\mu > 0$ is still valid. At this point, it is unclear whether $\mu_j \geq 0$ when $m > 1$. The following result follows from repeated differentiation.

Lemma 2. *Assume that $\mu_j \geq 0$ for all $j = 1, \dots, m$. Then the following statements hold:*

*(i) We have that $\omega(\lambda + \sum_{j=1}^m \mu_j l_{a_j}(\mathbf{x}|\mathbf{a}))$ is Δ-antitone in $\mathbf{x}$ if $l_{a_j}(\mathbf{x}|\mathbf{a})$ is Δ-antitone in $\mathbf{x}$ for all $j = 1, \dots, m$, and $\omega$ is $n$-antitone.*

*(ii) We have that $\omega(\lambda + \sum_{j=1}^m \mu_j l_{a_j}(\mathbf{x}|\mathbf{a}))$ is $\Delta^2$-antitone in $\mathbf{x}$ if $l_{a_j}(\mathbf{x}|\mathbf{a})$ is $\Delta^2$-antitone in $\mathbf{x}$ for all $j = 1, \dots, m$, and $\omega$ is $2n$-antitone.*

Example 1 (Continued). Note that whenever $\beta \leq \frac{1}{2}$, all three utility functions in Example 1 leads to $\omega$ functions that are $i$-antitone for *any* $i \geq 1$. Incidentally, all Jewitt's (1988, p. 1183) examples of distribution functions also have the feature that $l_a(x|a)$ is $i$-antitone for any $i \geq 1$. ◊

In the univariate case, a function is nondecreasing if and only if it is Δ-antitone. However, these concepts—nondecreasing and Δ-antitone—are distinct in the multivariate case. Similarly, any univariate function is increasing and concave if and only if it is $\Delta^2$-antitone, but this equivalence also fails in the multivariate case. These observations yield the first indication that there may be several ways to extend one-signal results based on FOSD or SOSD into higher dimensions. As emphasized in Section 5, if $m = 1$ and MLRP holds, then the likelihood ratio is automatically Δ-antitone in the important case where all signals are independently distributed.

### 4.2 *Multivariate FOSD and related stochastic orders*

As in the existing FOA literature, assume in the remainder of this section and the next two that $a$ is one-dimensional, or $m = 1$. The focus is thus on multiple signals, or $n \geq 2$. To begin, Müller and Stoyan (2002) make the following observation about extending the common stochastic orders from a univariate setting to a multivariate environment. Specifically, comparing two distribution functions, $G$ and $H$, there are three equivalent definitions of FOSD in the univariate setting, namely, (i) $G$ is preferred to $H$ for all nondecreasing utility function, (ii) $G(x) \leq H(x)$ for all $x$, and (iii) $\overline{G}(x) \geq \overline{H}(x)$ for all $x$. The point is that none of these definitions are equivalent when there are multiple signals. Consequently, there are three plausible ways to extend FOSD, which leads to the following definitions:

1. The function $G$ *first-order stochastically dominates* $H$ if $G$ is preferred to $H$ for all nondecreasing utility functions.

2. The function $G$ *dominates* $H$ *in the lower orthant order* if $G(\mathbf{x}) \leq H(\mathbf{x})$ for all $\mathbf{x}$.

3. The function $G$ *dominates* $H$ *in the upper orthant order* if $\overline{G}(\mathbf{x}) \geq \overline{H}(\mathbf{x})$ for all $\mathbf{x}$.

Using Conlon's (2009a) notation and terminology, let $\mathbf{E}$ be an increasing set. A set is increasing if $\mathbf{x} \in \mathbf{E}$ and $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \in \mathbf{E}$. It is well known that an equivalent definition of FOSD is that $G$ has more probability mass in all increasing sets than $H$ does; see Müller and Stoyan (2002, Theorem 3.3.4). Thus, since the upper orthants and the complements of the lower orthants are increasing sets, FOSD is stronger than the orthant orders. However, all three orders can be used to derive separate multi-signal justifications of the FOA.

Returning to the principal–agent model at hand, let

$$P(\mathbf{x} \in \mathbf{E}|a) = \int_{\mathbf{x} \in \mathbf{E}} f(\mathbf{y}|a)\, d\mathbf{y}$$

denote the probability that the vector of signals is in the set $\mathbf{E}$, given $a$. Let

$$P^L(\mathbf{x} \in \mathbf{E}|a, a^*) = P(\mathbf{x} \in \mathbf{E}|a^*) + (a - a^*)P_a(\mathbf{x} \in \mathbf{E}|a^*)$$

denote the counterpart in the auxiliary problem. Conlon (2009a) proposes a *concave increasing-set probability* (CISP) condition, specifically that $P_{aa}(\mathbf{x} \in \mathbf{E}|a) \leq 0$ for all increasing sets and all $a \in [\underline{a}, \overline{a}]$. The CISP condition implies that $P^L(\mathbf{x} \in \mathbf{E}|a, a^*) \geq P(\mathbf{x} \in \mathbf{E}|a)$ for all $a \in [\underline{a}, \overline{a}]$. In other words, $F^L(\mathbf{x}|a, a^*)$ first-order stochastically dominates $F(\mathbf{x}|a)$. Hence, the expected payoff in the auxiliary problem is greater than in the original problem as long as the FOA contract is monotonic, as continues to be the case as long as the (multivariate) MLRP holds. This explains Conlon's (2009a, Proposition 4) extension of Rogerson's conditions.

In practice, it may be hard to verify CISP. However, CISP can be weakened. Recall that the lower orthant order is weaker than FOSD. Moreover, it can also be expressed as an integral stochastic order. Specifically, it can be shown that $G$ dominates $H$ in the lower orthant order if and only if $G$ is preferred to $H$ for all $\Delta$-antitone utility functions (Müller and Stoyan 2002, Theorem 3.3.15). Although I do not pursue it here, the upper orthant order can also be used to form the basis of a new justification of the FOA; see Kirkegaard (2013). Those justifications necessitate that $\omega$ is convex.

Lemma 2 provides conditions under which the FOA contract gives rise to a $\Delta$-antitone utility function. Moreover, if $F_{aa}(\mathbf{x}|a) \geq 0$, then $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the lower orthant order. The condition that $F_{aa}(\mathbf{x}|a) \geq 0$ is henceforth referred to as the lower orthant convexity condition (LOCC).[9]

Definition 3 (LOCC). The term $F(\mathbf{x}|a)$ satisfies the *lower orthant convexity condition* (LOCC) if $F_{aa}(\mathbf{x}|a) \geq 0$ for all $\mathbf{x}$ and all $a$.

[9]Conlon (2009a, 2009b) considers a special case with two variables (see below). In this case, the orthant is a quadrant, and he thus defines the lower quadrant convexity condition (LQCC).

LOCC and CISP coincide in the one-signal case, where they collapse to the CDFC. In the multivariate case, however, CISP implies LOCC.

Proposition 3. *Assume the second-best action is in* $(\underline{a}, \overline{a})$. *Then the FOA is valid if* $F_{aa}(\mathbf{x}|a) \geq 0$ *for all* $\mathbf{x}$ *and all* $a$ *(LOCC),* $l_a(\mathbf{x}|a)$ *is* $\Delta$*-antitone in* $\mathbf{x}$ *for all* $a$*, and* $\omega$ *is* $n$*-antitone.*

Proof. The inequality $F_{aa}(\mathbf{x}|a) \geq 0$ implies that $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the lower orthant order. Hence, expected payoff in the auxiliary problem is higher than in the original problem as long as utility is $\Delta$-antitone. The remaining conditions ensure this is the case, since they allow Lemma 2 to be invoked. □

Jewitt (1988, Theorem 2) reports a special case with $n = 2$ independent signals. Conlon (2009a, 2009b) supplies the following generalization that allows correlated signals.

Assume there are two signals, and that the likelihood ratio is increasing and submodular in the two signals. Then Conlon (2009b) proves the FOA is valid if $F_{aa}(\mathbf{x}|a) \geq 0$. Submodularity means that the cross-partial derivative is nonpositive. Thus, with $n = 2$, $l_a(\mathbf{x}|a)$ is $\Delta$-antitone. In other words, this result is a special case of Proposition 3. However, Conlon (2009a) concludes that "it is not clear how to extend this beyond the two-signal case." The resolution comes from the observation that the submodular order and the lower orthant order coincide in the bivariate case.

Recall that Rogerson views the univariate CDFC as representing a type of "stochastic diminishing returns to scale"; the probability that the outcome exceeds any given threshold is concave in $a$. CISP has a similar interpretation in the multivariate case. It turns out that LOCC can be given a like interpretation, once it is taken into account that more structure is imposed on the payoffs in this case. To compare CISP and LOCC, consider first the following somewhat extreme increasing function. Fix some increasing set $\mathbf{E}$. The utility function $u_{\mathbf{E}}$ gives utility 1 if $x \in \mathbf{E}$ and 0 otherwise. Consistent with decreasing returns to scale, CISP says that the probability of a good outcome is concave in $a$. An extreme form of a $\Delta$-antitone utility function is the following. Fix some vector $\widehat{\mathbf{x}}$, and let $O$ denote the set $O = \{\mathbf{x}|\mathbf{x} \leq \widehat{\mathbf{x}}\}$, i.e., $O$ is the orthant that lies below $\widehat{\mathbf{x}}$. The utility function $u_O$ gives 0 utility if $\mathbf{x} \in O$ and utility 1 otherwise. LOCC amounts to saying that the probability of a good outcome is concave in $a$. Thus, given the structure of the contract, both CISP and LOCC capture a kind of "stochastic diminishing returns to scale." In the latter case, the assumption that $l_a(\mathbf{x}|a)$ is $\Delta$-antitone is key to establishing the correct structure. An interpretation of this assumption is provided in the Appendix.

### 4.3 *Multivariate SOSD and related stochastic orders*

Among the ingredients in Jewitt's (1988, Theorem 3) second set of conditions and Conlon's (2009a, Proposition 2) extension thereof are the assumptions that $l_a(\mathbf{x}|a)$ is increasing and concave in $\mathbf{x}$ and that $\omega$ is increasing and concave. These assumptions imply that $v(w(\mathbf{x}))$ is increasing and concave in $\mathbf{x}$, thus pointing in the direction of SOSD.

As in the univariate case, the distribution function $G$ second-order stochastically dominates $H$ if the former is preferred to the latter for any increasing and concave multivariate utility function.

However, to close the proof, Jewitt and Conlon depart from the Mirrlees formulation (where $F(\mathbf{x}|a)$ summarizes the technology). Instead, they utilize the state-space formulation of the principal–agent model. This change in modelling approach can be understood by the fact that it is *impossible* to express SOSD with a set of conditions directly on $F(\mathbf{x}|a)$ in the multivariate case; see Müller and Stoyan (2002, p. 98).

Other stochastic orders are better suited for the Mirrlees formulation. In particular, *G dominates H in the lower orthant-concave order* if

$$\int_{\underline{x}_1}^{x_1} \cdots \int_{\underline{x}_n}^{x_n} G(y_1, \ldots, y_n)\, dy_n \cdots dy_1 \leq \int_{\underline{x}_1}^{x_1} \cdots \int_{\underline{x}_n}^{x_n} H(y_1, \ldots, y_n)\, dy_n \cdots dy_1 \quad \text{for all } \mathbf{x},$$

as defined in Shaked and Shanthikumar (2007). Denuit and Mesfioui (2010) examine this and related stochastic orders. It can be shown that if $G$ dominates $H$ in the lower orthant-concave order, then $G$ is preferred to $H$ for any $\Delta^2$-antitone utility function. The following proposition then follows from the usual logic. Unsurprisingly, the condition that $\int_{\mathbf{y}\leq\mathbf{x}} F_{aa}(\mathbf{y}|a)\, d\mathbf{y} \geq 0$ for all $\mathbf{x}$ and all $a$ plays a key role. I refer to this condition as the cumulative lower orthant convexity condition (CLOCC).

Definition 4 (CLOCC). The function $F(\mathbf{x}|a)$ satisfies the *cumulative lower orthant convexity condition* (CLOCC) if $\int_{\mathbf{y}\leq\mathbf{x}} F_{aa}(\mathbf{y}|a)\, d\mathbf{y} \geq 0$ for all $\mathbf{x}$ and all $a$.

CLOCC implies that $F^L$ dominates $F$ in the lower orthant-concave order. The interested reader is directed to Kirkegaard (2013) for an additional justification of the FOA based on the so-called upper orthant-convex order.

Proposition 4. *Assume the second-best action is in* $(\underline{a}, \overline{a})$. *Then the FOA is valid if* $\int_{\mathbf{y}\leq\mathbf{x}} F_{aa}(\mathbf{y}|a)\, d\mathbf{y} \geq 0$ *for all* $\mathbf{x}$ *and all* $a$ *(CLOCC),* $l_a(\mathbf{x}|a)$ *is* $\Delta^2$*-antitone in* $\mathbf{x}$ *for all* $a$*, and* $\omega$ *is* $2n$*-antitone.*

Jewitt's one-signal conditions imply that the univariate function $v(w(x))$ is concave. In the univariate case, this is equivalent to saying that $v(w(x))$ has a negative second derivative. This equivalence does not hold in higher dimensions, however. Thus, to generalize into higher dimensions one could either pursue concavity of $v(w(\mathbf{x}))$ *or* pursue conditions on the sign of the second derivatives (as in Section 4.1). The former leads to SOSD and Conlon's CISP condition. The latter leads to the lower orthant-concave order and the CLOCC.

The justifications that I pursue require conditions only on the *sign* of higher-order derivatives. In comparison, multivariate concavity requires conditions on the *relative magnitude* of various second derivatives. In this sense multivariate concavity is a messier concept. This contrast explains the difference in tractability between Conlon's result and Proposition 4.

## 5. The additive property

Compared to Conlon's justifications, Propositions 3 and 4 have the advantage that the conditions on $F$ are weaker and easier to check. It is a drawback that stronger assumptions on $\omega$ are required, but Example 1 illustrates that this may be a small price to pay. However, the new justifications also impose possibly less transparent conditions on the cross-partial derivatives of $l_a(\mathbf{x}|a)$. The Appendix contains an interpretation of these conditions. Nevertheless, there are cases where these conditions are less restrictive (their dimensionality is reduced), and particularly appealing environments where they are not at all restrictive. For concreteness, I begin with the latter environments before deriving the more general result.

### 5.1 *Independent signals*

Assume that all signals are independent. Thus, $f(\mathbf{x}|a) = \prod_{i=1}^n f^i(x_i|a)$, where $f^i(x_i|a)$ denotes the marginal density of the $i$th signal. It follows that $l_a(\mathbf{x}|a)$ is separable, or

$$l_a(\mathbf{x}|a) = \sum_{i=1}^{n} \frac{f_a^i(x_i|a)}{f^i(x_i|a)} = \sum_{i=1}^{n} l_a^i(x_i|a),$$

where $l_a^i(x_i|a)$ is the likelihood ratio of the $i$th signal. Thus, all cross-partial derivatives are zero. Consequently, the restriction that $l_a(\mathbf{x}|a)$ is $\Delta$-antitone has no bite beyond MLRP. Moreover, it is known that $F_a^i(x_i|a) \leq 0$ when $l_a^i(x_i|a)$ is monotonic in $x_i$. Assume now that each independent signal satisfies the CDFC. Since $F^i$ is then decreasing and convex in $a$, it follows that $F(\mathbf{x}|a) = \prod F^i(x_i|a)$ is also convex in $a$. Hence, if each signal distribution satisfies Rogerson's assumptions (MLRP and CDFC), then the joint distribution satisfies LOCC. Proposition 3 can now be invoked.

Corollary 1. *Assume there are $n \geq 2$ independent signals, with distribution functions $F^i(x_i|a)$ and likelihood ratio $l_a^i(x_i|a)$, $i = 1, 2, \dots, n$. Assume the second-best action is in $(\underline{a}, \overline{a})$. Then the FOA is valid if the following statements hold:*

*(i) Each signal satisfies Rogerson's conditions; $F_{aa}^i(x_i|a) \geq 0$ and $l_{ax}^i(x_i|a) \geq 0$ for all $i = 1, 2, \dots, n$.*

*(ii) The function $\omega$ is $n$-antitone.*

As alluded to earlier, Corollary 1 generalizes Jewitt's (1988) Theorem 2, which assumes two independent signals. Similarly, Proposition 4 can be used to justify the FOA when each independent signal satisfies Jewitt's univariate assumptions.

Corollary 2. *Assume there are $n \geq 2$ independent signals, with distribution functions $F^i(x_i|a)$ and likelihood ratio $l_a^i(x_i|a)$, $i = 1, 2, \dots, n$. Assume the second-best action is in $(\underline{a}, \overline{a})$. Then the FOA is valid if the following statements hold:*

*(i) Each signal satisfies Jewitt's one-signal conditions; $\int_{\underline{x}_i}^{x_i} F_{aa}^i(y|a)\,dy \geq 0$, $l_x^i(x_i|a) \geq 0$, and $l_{axx}^i(x_i|a) \leq 0$ for all $i = 1, 2, \dots, n$.*

*(ii) The function $\omega$ is $2n$-antitone.*

Together, Corollary 1 and Corollary 2 offer an argument in favor of multi-signal conditions based on the orthant orders over conditions based on the more demanding multivariate notions of FOSD and SOSD, like Jewitt's Theorem 3 or Conlon's (2009a) propositions. In fact, Conlon (2009a) makes the point that if the $n$ signals are independent and each satisfies CDFC, then the joint distribution function may nevertheless fail the CISP condition. Conlon explains this failure by noting that "with many signals, the principal tends to become very well informed about the agent's action and, even in the one-signal case, [Rogerson's condition] must fail when the signal becomes very accurate."[10] While the logic is compelling, the above corollaries establish that the FOA can be justified as long as each signal satisfies standard assumptions. Thus, the FOA may be valid with a multitude of inaccurate independent signals even if they provide very precise information when combined.

There are two ways to think about integral stochastic orders: in terms of the relationship between the random variables or in terms of the generator of the stochastic order. Thus, there are also two ways to think about the difference between Conlon's results and the results in the present paper. First, LOCC requires only that $1-F(\mathbf{x}|a)=1-\prod F^i(x_i|a)$ is concave in $a$. This follows automatically from the properties that $F^i(x_i|a)$ is decreasing and convex in $a$ for all $i$. Note that $1-\prod F^i(x_i|a)$ describes the probability of a realization in one particular type of increasing set, namely the complement to the lower orthant. In contrast, CISP imposes a condition on all increasing sets. Consider, for instance, the probability that the realization is in the upper orthant, $\prod(1-F^i(x_i|a))$. Conlon's (2009a) observation is that this need not be concave in $a$ even if $(1-F^i(x_i|a))$ is concave in $a$ for all $i$.

The second way to contrast the results starts by noting that Conlon relies only on the weak property that $v(w(\mathbf{x}))$ is monotonic. In contrast, the new justifications rely on finer properties of how $v(w(\mathbf{x}))$ is structured. Specifically, $v(w(\mathbf{x}))$ is $\Delta$-antitone, which requires not only that $v(w(\mathbf{x}))$ is monotonic but also that the cross-partial derivatives of $v(w(\mathbf{x}))$ obey a sign restriction. As explained at the beginning of this section, independence implies that the latter is satisfied once the relatively weak assumption that $\omega$ is $n$-antitone is added. Exploiting the extra structure on $v(w(\mathbf{x}))$ then means that the conditions on $F(\mathbf{x}|a)$ can be relaxed.

In summary, the structure of $v(w(\mathbf{x}))$ determines which types of increasing sets need to be considered. Future research may attempt to make more inferences regarding the structure of $v(w(\mathbf{x}))$. This will make it clearer which types of increasing sets are potentially problematic and thus help determine which kinds of concavity assumptions are required.

### 5.2 *A general additive property*

To generalize Corollaries 1 and 2, consider two non-overlapping vectors of signals, $\mathbf{z}$ and $\mathbf{y}$, with $\mathbf{x}=(\mathbf{z},\mathbf{y})$. As in Holmström (1979), one could for example think of $\mathbf{z}$ as being

[10] In fact, it is easy to find examples where CISP fails with as little as $n=2$ i.i.d. signals. Thus, Corollaries 1 and 2 are of value even when $n$ is small.

signals or results that are directly payoff relevant to the principal, and of $\mathbf{y}$ as signals whose only function is to reduce the agency costs.

There are $n^z$ signals in $\mathbf{z}$ and $n^y$ signals in $\mathbf{y}$, with $n^z + n^y = n$. The two vectors have distribution functions $G(\mathbf{z}|a)$ and $H(\mathbf{y}|a)$, respectively. Assume that the two vectors of signals are independent of each other, with joint distribution $F(\mathbf{x}|a) = G(\mathbf{z}|a)H(\mathbf{y}|a)$. As another example, $\mathbf{z}$ and $\mathbf{y}$ could represent the answers to identical consumer surveys administered to two different consumers. Conditional on the fixed action—such as the effort the salesman has devoted to rehearsing the sales pitch both consumers are subjected to quite independently—it does not seem unreasonable to assume that $\mathbf{z}$ and $\mathbf{y}$ are themselves independent. The next section considers how to determine, e.g., the optimal number of consumers to survey.

Letting $g$ and $h$ denote the densities, independence implies that

$$l_a(\mathbf{x}|a) = l_a^g(\mathbf{z}|a) + l_a^h(\mathbf{y}|a),$$

where $l_a^g(\mathbf{z}|a) = g_a(\mathbf{z}|a)/g(\mathbf{z}|a)$ and $l_a^h(\mathbf{y}|a) = h_a(\mathbf{y}|a)/h(\mathbf{y}|a)$. Note that if $l_a^g(\mathbf{z}|a)$ and $l_a^h(\mathbf{y}|a)$ are $\Delta$-antitone, then so is $l_a(\mathbf{x}|a)$. To employ Proposition 3, however, it is also required that $F_{aa} \geq 0$. Unfortunately, this is not implied by assuming that $G$ and $H$ satisfy MLRP and LOCC. The reason is that in the multivariate case, MLRP does not seem to imply, e.g., $G_a \leq 0$. Thus, another assumption is required.

Definition 5 (NLOP). A distribution function $F(\mathbf{x}|a)$ has the *nonincreasing lower orthant probability* (*NLOP*) property if $F_a(\mathbf{x}|a) \leq 0$ for all $\mathbf{x}$ and all $a$.

The NLOP property is inspired by Conlon's *nondecreasing increasing-set probability* (NISP) property that $P_a(\mathbf{x} \in \mathbf{E}|a) \geq 0$ for all increasing sets $\mathbf{E}$.[11] Note that NISP implies NLOP. Conlon notes that NISP is satisfied whenever MLRP holds and the signals are affiliated (see the Appendix). Given that NLOP and LOCC for $G$ and $H$ guarantee that these are decreasing and convex in $a$, it follows that $F(\mathbf{x}|a)$ is convex as well (it satisfies LOCC).

Proposition 5. *Assume $G(\mathbf{z}|a)$ and $H(\mathbf{y}|a)$ are independent and each satisfies NLOP and LOCC. Assume $l_a^g(\mathbf{z}|a)$ and $l_a^h(\mathbf{y}|a)$ are $\Delta$-antitone and that $\omega$ is $n$-antitone. Finally, assume the second-best action is in $(\underline{a}, \overline{a})$. Then the FOA is valid not only if the principal has access to either $\mathbf{z}$ or $\mathbf{y}$, but also if he has access to both.*[12]

Proposition 5 can clearly be extended to any arbitrary number of non-overlapping vectors of signals. Technically, the proposition also implies Corollary 1. The reason is that independent signals are affiliated, and so MLRP implies NLOP. For completeness, the next result records the obvious generalization of Corollary 2.

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[11] As Conlon (2009a) remarks, NISP is in turn inspired by Sinclair-Desgagné's (1994) generalized stochastic dominance condition. Conlon (2009a) imposes NISP to handle a risk averse principal.

[12] The caveat to Propositions 5 and 6 as well as Corollaries 1 and 2 is that it is implicitly assumed that the FOA yields wages in the interior for all $n$ under consideration. As mentioned earlier, this is typically satisfied if $\overline{u}$ is large enough.

Proposition 6. *Assume $G(\mathbf{z}|a)$ and $H(\mathbf{y}|a)$ are independent and each satisfies NLOP and CLOCC. Assume $l_a^g(\mathbf{z}|a)$ and $l_a^h(\mathbf{y}|a)$ are $\Delta^2$-antitone and that $\omega$ is $2n$-antitone. Finally, assume the second-best action is in $(\underline{a}, \overline{a})$. Then the FOA is valid not only if the principal has access to either $\mathbf{z}$ or $\mathbf{y}$, but also if he has access to both.*

## 6. Multi-signal monitoring

A common approach to monitoring is to hold fixed the number of signals. For instance, Kim (1995) and Demougin and Fluet (2001) develop ways to rank the informativeness of different information systems. Allgulin and Ellingsen (2002) and Holmström and Milgrom (1994) provide somewhat different models in which the principal can invest in the accuracy of the information system. Both models predict that the agent is induced to work harder when it becomes cheaper to improve the information system.

Alternatively, Holmström (1979) examines when the agency costs diminish as more signals are added, and how the optimal contract changes for a given action.[13] Conceptually, the number of signals is thus endogenous, and determined by their costs and their ability to reduce agency costs.

It should also be clear that the principal is likely to induce a different action when the information system changes. However, Holmström (1979) does not analyze how the second-best action changes as more signals are acquired. I focus on that question here. For concreteness, I pursue the question in the context of a particularly tractable setup, which I refer to as the square root, independent tasks (SQIT) model. The SQIT model is sufficient to demonstrate a few interesting negative results. For instance, more monitoring may lead to an action further removed from the first best. First, however, the structure of the LEN and SQIT models are contrasted. They are compared more fully in the Appendix.

Example 2 (Linear-exponential-normal model). The static linear-exponential-normal (LEN) model assumes that the agent's action costs are monetary, and that utility of income is exponential (exhibiting constant absolute risk aversion). Bernoulli utility is thus $-e^{-r(w-c(\mathbf{a}))}$, $r > 0$. The agent's action, $\mathbf{a}$, determines the means of $n$ jointly normally distributed signals. The covariance matrix is independent of $\mathbf{a}$. The contract is restricted to be linear in the signals. See Holmström and Milgrom (1988, 1991, 1994) for applications. Holmström and Milgrom (1987) provide a dynamic foundation of the LEN model. ◊

Example 3 (Square root, independent tasks model). As before, assume Bernoulli utility is $v(w) - c(\mathbf{a})$. The SQIT model makes two additional assumptions. First, utility of income is $v(w) = 2\sqrt{w}$. Second, each dimension of the agent's action is an independent task, in the sense that it produces a vector of signals that is independent of the vectors of signals that are produced by other tasks. The second assumption is discussed and formalized in the next section. In this section it suffices to examine the single-task version of the model.

[13]Likewise, in the literature on contingent monitoring the principal can purchase another signal after having observed a first signal; see, e.g., Jewitt (1988, Section 5) who justifies the FOA in such a two-signal environment (Holmström also mentions contingent monitoring in passing).

A closed-form solution for the optimal contract can be derived (see the Appendix). In the single-task case, the expected cost to the principal of inducing $a$ is

$$K(a) = \left(\frac{\overline{u} + c(a)}{2}\right)^2 + \frac{c'(a)^2}{4V(a)},$$

where

$$V(a) = \int l_a(\mathbf{x}|a)^2 f(\mathbf{x}|a)\, d\mathbf{x}$$

is the Fisher information. Fisher information is a well known measure of how informative $\mathbf{x}$ is about $a$. Thus, the SQIT model allows a more flexible information structure than the LEN model. In particular, the informativeness of the signals may depend on $a$. This is not the case in the LEN model where the covariance matrix is independent of $a$. ◊

Assume that signals can be grouped into independent "blocks," with each block satisfying the assumptions in Propositions 5 or 6. Since $\omega$ is $i$-antitone for any $i$ in the SQIT model, the FOA is valid regardless of the number of blocks. Each block could represent, e.g., a unique consumer survey or referee report or the multidimensional results of a test of a product sample. The number of blocks is endogenous.

To fix ideas, assume the principal cares directly only about the first block of signals, which is observed for free (it may represent, e.g., revenue). The Fisher information of the $i$th signal block is $V^i(a) > 0$. By independence, the Fisher information of the entire information system when the principal has acquired the first $n$ blocks is then $V(a,n) = \sum_{i=1}^n V^i(a)$. Recall that the higher is $V(a,n)$ , the more informative is the information system. This explains why $V(a,n)$ is increasing in $n$.

From Example 3, the principal's expected profit from implementing action $a$ with a total of $n$ signal blocks is

$$\Pi(a,n|t) = \left[B(a) - \left(\frac{\overline{u} + c(a)}{2}\right)^2\right] - \left[\frac{c'(a)^2}{4V(a,n)} + (n-1)t\right], \tag{8}$$

where $t > 0$ is the cost of acquiring another block of signals beyond the first. The first square bracket coincides with the principal's payoff under full information. Let $\pi(a)$ denote this term. Assuming $B(a)$ is concave, $\pi(a)$ is then concave. The last terms capture the agency and signal costs. For simplicity, assume that for any $n$, the principal's overall payoff function $\Pi(a,n|t)$ is single-peaked in $a$, with an interior maximum that is more profitable than inducing $\underline{a}$ with a fixed wage. Note that (8) is concave in $V$, reflecting decreasing returns to scale in information. As is intuitive, it can be shown that the optimal number of signals ($n$) weakly increases when information becomes cheaper ($t$ decreases). Thus, the comparative statics are the same in the hypothetical situation where $t$ decreases as in the situation where the principal is initially endowed with $n$ signals and then exogenously receives more.

Next, observe that if the agency and signal costs are independent of $a$ for all $n$, then the second-best action, $a^*$, in fact coincides with the first best, $a^{FB}$. This occurs for instance if $c'(a)$ and $V^i(a)$ are constant, for all $i$. Likewise, the second-best action falls below the first-best action if the last term in (8) is globally increasing in $a$. Since $c'(a)$ is (weakly) increasing, this must happen if $V(a,n)$ is decreasing in $a$, which in turn is guaranteed if $V^i(a)$ is a decreasing function for all $i$. If $c'(a)$ is strictly increasing, the same conclusion obtains even if $V(a,n)$ is constant in $a$. The Appendix provides sufficient conditions for $V^i(a)$ to be decreasing in $a$, but also presents examples where this is not the case.

The following examples explore comparative statics. Example 4 builds intuition for how $a^*$ changes with $n$ in the simplest setting. Example 5 establishes that the second-best action may be non-monotonic in the number of signals. Thus, the second best may move further away from the first best when an additional signal is acquired.[14]

Example 4. Assume that all signal blocks are identically distributed. Thus, $V(a,n) = nV^1(a)$. It is easy to see that $\Pi(a,n)$ is supermodular, i.e., $\Pi_{an}(a,n) \geq 0$, if $(V^1)'(a) \leq 0$. Thus, $a^*$ is increasing in $n$. It follows that as $t$ decreases, $a^*$ increases. In summary, $a^* < a^{FB}$ but cheaper monitoring leads the principal to induce a higher action, such that $a^*$ approaches $a^{FB}$, from below. If $(V^1)'(a) > 0$ and $c''(a) = 0$, then $\Pi_{an}(a,n) < 0$. Hence, $a^*$ declines with $t$ but once again approaches $a^{FB}$, this time from above. In either case, when $n$ is small, $a^*$ is chosen in large part to control agency costs by ensuring that signals are very informative ($V(a^*,n)$ is large compared to $V(a^{FB},n)$). As $n$ increases, the quality of the information system increases and agency costs are lower, for any $a$. More importantly, the value of marginally improving the information system by manipulating $a$ is lowered as well.[15] Thus, distorting the second best away from the first best is less attractive. Stated differently, it is cheaper for the principal to move the action closer to the first best as $n$ increases. Note that contrary to Holmström and Milgrom (1994), who use a version of the LEN model, the second best may increase or decrease as monitoring becomes cheaper. ◊

Example 5. Assume now that all signal blocks after the first are identically distributed. Thus, $V(a,n) = V^1(a) + (n-1)V^2(a)$. Moreover, assume that $c'(a) = c$ and $V^1(a) = V^1$ are constant but allow $V^2(a)$ to differ from $V^1$ and to depend on $a$. Then, without any additional monitoring ($n = 1$), the principal induces the first-best action, $a^{FB}$. When $n > 1$, however,

$$\Pi_a(a,n) = \pi'(a) + c^2 \frac{(V^2)'(a)}{4} \frac{(n-1)}{V(a,n)^2},$$

[14]The second best need not converge to the first best as the number of signals approaches infinity. Imagine that the informativeness of successive signals is rapidly declining. For instance, if $F^i(x_i|a) = 1 - e^{-x_i a^{-1/i}}$, $x_i \geq 0$, $a > 0$, then $V^i(a) = 1/a^2 i^2$ and $V(a,n) \to \frac{1}{6}\pi^2/a^2 < \infty$ as $n \to \infty$. Full information is never approached, and the second best is bounded away from the first best. An implication is that justifications of the FOA are relevant even when $n$ is arbitrarily large; it is not the case that the agency problem necessarily disappears as $n \to \infty$.

[15]The agency costs, $\frac{1}{4}c'(a)^2/(nV^1(a))$, are decreasing in $V$, but at a slower rate the higher $n$ is.

and it follows that as long as $(V^2)'(a) \neq 0$, it is optimal for the principal to induce an action different from the first best. Nevertheless, note that the last term in $\Pi_a(a, n)$ vanishes as $n \to \infty$. That is, in the limit, $a^*$ converges once again to $a^{FB}$. Hence, $a^*$ is nonmonotonic in $n$. A corollary is that there must be a range of $n$ over which $a^*$ moves away from $a^{FB}$ as $n$ increases.[16] ◊

## 7. Extension: Pure multitasking

This section outlines how the new approach can be used to derive the first justifications of the FOA when the action is multidimensional. Details are given in the Appendix.

A major challenge to extending the FOA to many-dimensional actions is to sign the multipliers. Thus, I further specialize the model by assuming a certain independence between the different dimensions of the agent's action. Conceptually, the different dimensions of $\mathbf{a}$ can be interpreted as representing distinct tasks. Thus, define a *pure multitasking environment* to be any environment where $n \geq m$, in which $F(\mathbf{x}|\mathbf{a})$ can be written as

$$F(\mathbf{x}|\mathbf{a}) = G^1(\mathbf{x}^1|a_1)G^2(\mathbf{x}^2|a_2)\cdots G^m(\mathbf{x}^m|a_m),$$

where each $G^j$ is a distribution function, and where there is no overlap between the $\mathbf{x}^j$s, such that $\mathbf{x} = (x_1, \dots, x_n) = (\mathbf{x}^1, \dots, \mathbf{x}^m)$. Let $n^j$ denote the number of signals in $\mathbf{x}^j$, with $\sum n^j = n$. Tasks are isolated from one another in the sense that task $j$ is associated with a particular set of signals, $\mathbf{x}^j$, the distribution of which is independent of effort on other tasks. From the agent's point of view, the tasks are dependent only through $c(\mathbf{a})$. Let $g^j$ denote the density of $G^j$.

The pure multitasking environment implies a useful separability between tasks, with

$$l_{a_j}(\mathbf{x}|\mathbf{a}) = \frac{\partial \ln f(\mathbf{x}|\mathbf{a})}{\partial a_j} = \frac{g^j_{a_j}(\mathbf{x}^j|a_j)}{g^j(\mathbf{x}^j|a_j)}.$$

Thus, in the following discussion $l_{a_j}(\mathbf{x}|\mathbf{a})$ is denoted $l^j_{a_j}(\mathbf{x}^j|a_j)$. The simple structure of the model along with the additional assumption that $G^j(\mathbf{x}^j|a_j)$ satisfies NISP make it possible to prove that $\mu_j > 0$ and establish conditions under which the FOA is valid. Recall that $G^j(\mathbf{x}^j|a_j)$ automatically satisfies NISP if $n^j = 1$ and MLRP is satisfied. Thus, in the Appendix I show that the FOA is valid if, for all $j$, $G^j(\mathbf{x}^j|a_j)$ satisfies NISP, $l^j_{a_j}(\mathbf{x}^j|a_j)$ is $\Delta$-antitone in $\mathbf{x}^j$, $F(\mathbf{x}|\mathbf{a})$ satisfies the multidimensional version of LOCC (i.e., $F(\mathbf{x}|\mathbf{a})$ is convex in $\mathbf{a}$), and $\omega$ is $n$-antitone. Justifications relying on multidimensional versions of CISP and CLOCC are also straightforward. A counterpart to the additive property is also possible under which the FOA remains valid as more and more independent tasks are added. The SQIT model is reviewed next.

[16] A referee pointed out that it is easy to construct such examples even in simple models with three action levels and two outcomes. Assume that with just one signal, the highest action is both first best and second best. If an additional signal is uninformative at the highest action but very informative at the intermediate action, the second best may move away from the first best. It is the same logic that is at play in Example 5. However, the example illustrate the richness of the SQIT model and its tractability even with a continuum of actions and outcomes.

Example 6 (Square-root, independent tasks model, continued). Consider a pure multitasking environment with $v(w) = 2\sqrt{w}$. Assuming the FOA is valid, the multipliers can be derived in closed form, with $\lambda = \frac{1}{2}(\overline{u} + c(\mathbf{a}^*))$ and $\mu_j = \frac{1}{2}c_j(\mathbf{a}^*)/V_j(a_j^*) > 0$, where

$$V_j(a_j^*) = \int (l_{a_j}^j(\mathbf{x}^j|a_j^*))^2 g^j(\mathbf{x}^j|a_j^*)\,d\mathbf{x}$$

is the Fisher information, measuring how informative $\mathbf{x}^j$ is about $a_j$. Note that $\mu_j > 0$ even without assuming NISP. Thus, in the Appendix I prove that the FOA is valid in this model even in cases where NISP is violated.

The contract can be characterized in closed form. The agent's utility of income is

$$v(w(\mathbf{x})) = \overline{u} + c(\mathbf{a}^*) + \sum_{j=1}^{m}\left(\frac{c_j(\mathbf{a}^*)}{V_j(a_j^*)} l_{a_j}^j(\mathbf{x}^j|a_j^*)\right),$$

which is separable in the different $\mathbf{x}^j$s. Special cases has been considered in the existing literature. In particular, Jewitt et al. (2008) characterize the solution when $n = m = 1$. Holmström (1979) also considers an example with $n = m = 1$ but he imposes the additional assumption that $x$ is exponentially distributed.

The cost of implementing $\mathbf{a}^*$ can be computed in the SQIT model. It depends only on $\overline{u}$, $c(\mathbf{a}^*)$, and $V_j(a_j^*)$ (see the Appendix). Thus, it is feasible to do various comparative statics exercises in this model. ◊

The Appendix demonstrates that the SQIT model can be used to obtain new economic insights and to challenge conclusions based on the LEN model normally used to examine multitasking. To stage the comparison, note that the reciprocal of a signal's variance is a natural measure of its informativeness in the LEN model. The SQIT model has a similarly clean measure of informativeness, namely Fisher information. However, a key difference between the two models is that the LEN model assumes that the covariance matrix is independent of **a**, whereas the SQIT model allows Fisher information to depend on **a**. As in the examples in Section 6, this property produces an incentive—which is absent in the LEN model—for the principal to push the agent toward more informative actions or tasks.

Holmström and Milgrom (1988) remark that if the agent's cost function is symmetric in the LEN model, then the agent will be induced to work harder on the more informative task. This result now seems to be commonly accepted, as evidenced by the explanation in, e.g., Dixit (1997), who states, "[c]onsidering each task in isolation, one with a more accurately observed outcome would have a higher-powered incentive because the outcome is a better indicator of the effort one wants to motivate."

The logic in the above quote implicitly relies on the accuracy of the signals being independent of the action. However, the principal's incentive to induce effort on various tasks changes if and when the quality of the information depends on task levels. To be clear, a similar effect is also present in the standard single-task model. However, its significance is amplified in a multi-task setting through the additional decision of how to distribute total effort among the set of tasks. In fact, this effect alone is sufficient to alter some of the equilibrium properties. For instance, the agent may be induced to work harder on a less informative task. This may occur if the informativeness of the latter task is very responsive to changes in effort. Likewise, the agent may be induced to work on two different tasks even if they are perfect substitutes. Such conclusions cannot be obtained in the LEN model.

## 8. Conclusion

A new approach to the moral hazard problem has been suggested. The approach is based on a reformulation of the problem that allows standard results from choice under uncertainty to be invoked so as to prove new and old results on the FOA.

First, the new approach unifies the proofs of the existing one-task justifications of the FOA. Second, the approach is utilized to derive new justifications of the FOA in economic environments involving multiple signals or multiple tasks. The new multi-signal justifications are in some ways more tractable than the existing ones. One distinct advantage is that they are robust to the inclusion of more independent signals. More generally, the paper's new additive property establishes conditions under which the problem is in a sense scale-free. A counterpart to the additive property holds in models with pure multitasking. Specifically, the FOA remains valid under familiar conditions as more tasks are added, provided all tasks are stochastically independent. These appear to be the first multi-task justifications of the FOA.

To illustrate the usefulness of the new justifications of the FOA I examine two questions that would be hard to answer without these new justifications. The first considers how many costly signals the principal should acquire and how the second-best action depends on that number. The second question addresses how to optimally induce the agent to distribute his effort among a distinct set of tasks. To answer these questions I present and analyze the SQIT model. The SQIT model remains tractable even as an arbitrary numbers of signals or tasks are added. Using this model it is shown that the change in the second-best action is not unambiguous when more signals are added. In the case of multitasking, the equilibrium properties are different than those of the popular LEN model. For example, it may be optimal to induce the agent to work hardest on the least informative task (see the Appendix). The SQIT model thus cautions that the LEN model is not robust. The root cause is easily identified: informational quality in the LEN model is independent of how total effort is distributed among tasks. The SQIT model is richer because it allows the quality of the information system to depend on effort. Thus, it is hoped that the SQIT model may in the future function as an alternative laboratory in which the consequences of moral hazard can be explored. It is especially well suited to applications in which the quality of the information system is manipulable.

Space considerations precludes more applications of the methodology in this paper. Kirkegaard (2015) uses the new multi-task justifications of the FOA as a stepping stone to analyze an extension of the canonical moral hazard model. Specifically, it is assumed that the agent faces some noncontractible uncertainty. For instance, the agent may earn income outside of the principal–agent relationship or face uncertainty regarding, e.g., his health status. The agent performs two "tasks." The first task is the job he performs on behalf of the principal, while the second represents his pursuit of private rewards. The FOA can be justified in this setting as well.

This paper has emphasized the ability of the new approach to justify the FOA and explored some of the implications of these new justifications. However, the methodology has uses beyond the FOA. For instance, Kirkegaard (2014) provides the first complete solution to the moral hazard problem under the so-called spanning condition. Conveniently, it is possible to identify exactly which "nonlocal" incentive compatibility constraints are binding. Thus, the model is well suited to exploring comparative statics when the FOA is not valid. In short, the new approach may be a useful tool more generally.

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