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A theory of stability in many-to-many matching markets

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We develop a theory of stability in many-to-many matching markets. We give conditions under which the setwise-stable set, a core-like concept, is nonempty and can be approached through an algorithm. The usual core may be empty. The setwise-stable set coincides with the pairwise-stable set and with the predictions of a non-cooperative bargaining model. The setwise-stable set possesses the conflict/coincidence of interest properties from many-to-one and one-to-one models. The theory parallels the standard theory of stability for many-to-one, and one-to-one, models. We provide results for a number of core-like solutions, besides the setwise-stable set.

KEYWORDS. Two-sided matching, cooperative game theory, core.

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1. INTRODUCTION

Consider a collection of firms and consultants. Each firm wishes to hire a set of consultants, and each consultant wishes to work for a set of firms. Firms have preferences over the possible sets of consultants, and consultants have preferences over the possible sets of firms. This is an example of a "many-to-many" matching market. A matching is an assignment of sets of consultants to firms, and of sets of firms to consultants, so that firm f is assigned to consultant w if and only if w is also assigned to f. The problem is to predict which matchings can occur.

Many-to-many matching markets are understood less well than many-to-*one* markets, in which firms hire many workers, but each worker works for only one firm. The many-to-one market model seems to describe most labor markets, so why should one study many-to-many markets? There are two reasons.

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First, some important real-world markets are many-to-many. One example is firms/consultants. But the best-known example is probably the market for medical interns in the U.K. (Roth and Sotomayor 1990). This example is important because it works through a centralized matching mechanism. Further, the theory of many-to-one matchings has helped in understanding and shaping centralized matching mechanisms for medical interns in the U.S. (Roth and Peranson 1999). Another example is the assignment of teachers to high schools in some countries (35% of teachers in Argentina work in more than one school). The assignment of teachers to high schools is a clear candidate for a centralized solution guided by theory. Finally, one can view many-to-many matching as an abstract model of contracting between down-stream firms and up-stream providers.

Second, even a few many-to-many contracts can make a crucial difference, and most labor markets have at least a few many-to-many contracts. We present an example (Section 2.2) of a many-to-one market where the number of agents can be arbitrarily large and still one many-to-many contract changes the contracting outcome for all agents. In the U.S., 76% of total employment is in industries with 5% or more multiple jobholders.[1] If even a few multiple jobholders (many-to-many contracts) make an important difference, we need a many-to-many model to understand the bulk of the labor markets in the U.S.

We first give an overview of our solutions and results. Then we place our results in the related literature.

1.1 *Overview of solutions*

We argue that the core is a potentially problematic solution for many-to-many markets. One problem is that the core may not be *individually rational*, in the sense that there are core matchings where a firm—for example—would be better off firing some worker. Another problem, well-known in the literature, is that the core may not be *pairwise stable*, in the sense that there may be a firm f and a worker w that are not currently matched, but where w would like to work for f, and f would like to hire w.

The situation contrasts with one-to-one and many-to-one matching markets, in which the standard solution is the set of pairwise-stable and individually-rational matchings. In one-to-one and many-to-one markets, this standard solution coincides with the core.

We consider alternatives to the core. One alternative is the *setwise-stable set* of Roth (1984) and Sotomayor (1999): the set of individually-rational matchings that cannot be blocked by a coalition that forms new links only among its members, but may preserve its links to agents outside of the coalition. A second alternative is the *individually-rational core* (defined implicitly by Sotomayor (1999)): the set of individually-rational matchings that cannot be blocked using an individually-rational matching. A third alternative is the *pairwise-stable set*, described above.

[1]Source: 2002 Annual averages of employed multiple job holders by industry, Division of Labor Force Statistics, U.S. Bureau of Labor Statistics.

A fourth alternative is the *fix-point set*: matchings where each agent a is choosing her best set of partners, out of the set of potential partners who, given their current match, are willing to link to a. In a matching in the fix-point set, a firm, for example, might be better off with a different set of workers than the one she has, but this set must involve at least one worker who prefers not to add the firm to his list of employers. The definition of the fix-point set is circular; it is defined as the set of fixed points of a certain operator.

1.2 *Overview of results*

We consider two restrictions on agents' preferences. The first is *substitutability*, first introduced by Kelso and Crawford (1982) and used extensively in the matching literature. The second is a strengthening of substitutability that we call *strong substitutability*.

We explain the two hypotheses. Let f be a firm. Substitutability (Definition 3.4) of f's preferences requires: "if hiring w is optimal when the set of available workers is $\{w\} \cup S'$, and S is a subset of S', then hiring w must still be optimal when the set of available workers is $\{w\} \cup S$." Strong substitutability (Definition 6.2) requires: "if hiring w is optimal when the set of available workers is $\{w\} \cup S'$, and the firm prefers S' to S, then hiring w must still be optimal when the set of available workers is $\{w\} \cup S$." Substitutability requires that, if w is chosen from a given set of workers, she is chosen also from a smaller set of workers. Strong substitutability, on the other hand, says that, if w is chosen from a given set of workers, she is chosen also from a worse set of workers.

Strong substitutability is stronger than substitutability. But it is weaker than separability, and not stronger than responsiveness—two other assumptions used in matching theory (separability is used extensively also in social choice).

We now enumerate and briefly discuss our main results.

For economy of exposition, we present results as results on the fix-point set, which we denote by $\mathscr{E}(P)$. The implications for the other solutions should be clear at all times.

If preferences are substitutable, the fix-point set is nonempty, and we give an algorithm for finding a fix-point matching; the fix-point set equals the set of individually rational and pairwise-stable matchings; a basic non-cooperative bargaining game—firms propose to workers, then workers propose to firms—has the fix-point set as its set of subgame-perfect equilibrium outcomes; a matching in the fix-point set that is blocked (in the sense of the core) must be blocked in a "non-individually-rational way," through a coalition of agents that all have incentives to deviate from the block. Note that these results translate into results about the pairwise-stable set, as it equals the fix-point set.

If firms' preferences are substitutable, and workers' preferences are strongly substitutable, the fix-point set equals the set of setwise-stable matchings, and a matching in the fix-point set must be in the individually-rational core. Thus setwise-stable matchings exist, the individually-rational core is nonempty, and our algorithm finds a matching in the individually-rational core that is setwise stable. Our model is symmetric; the same results hold when firms' preferences are strongly substitutable and workers' preferences are substitutable.

If preferences are substitutable, the fix-point set has certain properties one can interpret as worker-firm conflict of interest and worker-worker (or firm-firm) coincidence of interest: the fix-point set is a lattice. That the fix-point set is a lattice implies that there is a "firm-optimal" fix-point matching—a matching that is simultaneously better for all firms, and worse for all workers, than any other matching in the fix-point set—and a "worker-optimal" matching—one that is best for all workers, and worse for all firms. Besides the lattice-structure, there are other conflict/coincidence of interest properties from one-to-one and many-to-one markets (Roth 1985). We extend these properties to many-to-many markets. If preferences are strongly substitutable, the lattice operations on the fix-point set are the standard lattice operations from one-to-one matching markets. The stated properties translate to the pairwise-stable, and setwise-stable, sets by their relation to the fix-point set under the various restrictions on preferences.

If firms' preferences are substitutable and workers' preferences are strongly substitutable, the theory of many-to-many matchings parallels the theory of many-to-one matchings: the setwise-stable set equals the pairwise-stable set, and the setwise-stable set is a nonempty lattice. In the standard many-to-one model, firms' preferences are substitutable and workers' preferences are trivially strongly substitutable. So our model encompasses standard many-to-one theory.

In sum, we give conditions (substitutability, strong substitutability) under which our alternatives to the core are nonempty and can be approached through an algorithm. We emphasize that—even under the strongest of our hypotheses—the core may be empty. The setwise-stable set, the fix-point set, and the pairwise-stable set are identical and possess a lattice structure. The setwise-stable set, the fix-point set, and the pairwise-stable set coincide with the outcomes of a simple non-cooperative bargaining model. We reproduce and extend conflict/coincidence of interest properties.

1.3 *Related literature*

Setwise stability was first defined by Roth (1984). Sotomayor (1999) emphasizes the difference between setwise stability, pairwise stability, and the core. Sotomayor (1999) presents examples where the setwise-stable set is empty; preferences in her examples are not strongly substitutable. Sotomayor (1999) refers to a definition of core that coincides with our individually-rational core. We are the first to prove positive results on the setwise-stable set and the individually-rational core, in particular that they are nonempty.

The previous literature on many-to-many matchings has results for the pairwise-stable set. Roth (1984) proved that, with substitutable preferences, the pairwise-stable set is nonempty, and there are firm- and worker-optimal pairwise-stable matchings. Blair (1988) proved that the pairwise-stable set has a lattice structure. A standard objection (Roth and Sotomayor 1988) to pairwise-stability is that it does not allow for more general coalitions. We show that, under our structure on preferences, allowing for more general coalitions does not make a difference. As by-products of our results, we reproduce Roth's and Blair's results on pairwise stability using fixed-point methods—similar methods have been used in matching contexts by Adachi (2000), Echenique and Oviedo (2004), Hatfield and Milgrom (2005), Roth and Sotomayor (1988), and Fleiner (2003).

Blair's lattice structure relies on a somewhat artificial order; under our strongest structure, we obtain a lattice structure with the standard order in one-to-one theory.

Roth (1985) discusses global conflict/coincidence of interest properties, beyond the lattice property of pairwise-stable matchings. We extend Roth's results to many-to-many matchings.

Recently, Sotomayor (2004) proved that, with responsive preferences, a mechanism that coincides with the one in Section 7, implements the pairwise-stable matchings. She also proves that, under a restriction on preferences she calls "maxmin," the pairwise-stable matchings are in the core. Sotomayor claims that one can modify her proof to show that pairwise-stable matchings are setwise stable.

In independent work, Konishi and Ünver (2005) show that a concept they call credible group stability is equivalent to pairwise stability. Credible group stability is similar in spirit (but logically unrelated) to our bargaining set (defined in Section 4.3). Konishi and Ünver require preferences to be responsive and satisfy a separability assumption; their model is neither more nor less general than our model.

A precedent to our results on a bargaining set is Klijn and Massó (2003). Klijn and Massó study Zhou's bargaining set for the one-to-one matching model. The bargaining set we propose is different from Zhou's bargaining set. We compare the two in Section 4.3.

Alcalde et al. (1998) and Alcalde and Romero-Medina (2000) prove that the core is implemented in certain many-to-one models by simple mechanisms, similar to the one we present in Section 7.

Alkan (2001, 2002) presents properties (including distributivity) of the lattice structure on pairwise-stable matchings, in many-to-many markets with additional structure on preferences. Martínez et al. (2004) present an algorithm that finds all the pairwise-stable matchings in a many-to-many matching market.

2. Motivating examples

We give two motivating examples. The first example shows that the core may be a problematic solution. The second example shows that many-to-many matchings can be important, even when only a small number of agents on one side of the market are allowed to match with more than one agent on the other side.

2.1 *A problem with the core*

Example 2.1. Suppose the set of workers is $W=\{w_1,w_2,w_3\}$ and the set of firms is $F=\{f_1,f_2,f_3\}$. Workers' preferences are

$$\begin{aligned} P(w_1):&\quad f_3, f_2f_3, f_1f_3, f_1, f_2\\ P(w_2):&\quad f_1, f_1f_3, f_1f_2, f_2, f_3\\ P(w_3):&\quad f_2, f_1f_2, f_2f_3, f_3, f_1. \end{aligned}$$

The notation means that w_1 prefers $\{f_3\}$ to $\{f_2,f_3\}$, $\{f_2,f_3\}$ to $\{f_1,f_3\}$, $\{f_1,f_3\}$ to $\{f_1\}$, and so on. We often omit braces ($\{\ldots\}$) when denoting sets. If $A\subseteq F$ is not listed it means that $\emptyset$ is preferred to A. Firms' preferences are

$$\begin{aligned} P(f_1):&\quad w_3, w_2w_3, w_1w_3, w_1, w_2\\ P(f_2):&\quad w_1, w_1w_3, w_1w_2, w_2, w_3\\ P(f_3):&\quad w_2, w_1w_2, w_2w_3, w_3, w_1. \end{aligned}$$

Consider the matching $\hat{\mu}$ defined by $\hat{\mu}(w_1)=\{f_2,f_3\}$, $\hat{\mu}(w_2)=\{f_1,f_3\}$, and $\hat{\mu}(w_3)=\{f_1,f_2\}$.

Note that $\hat{\mu}$ is a core matching: To make f_1—for example—better off than in $\hat{\mu}$, f_1 should hire only w_3, which would make w_2 hired only by f_3, so w_2 would be worse off. Now, if f_1 is in a blocking coalition $\mathscr{C}$, w_3 must be in $\mathscr{C}$. Then f_2 must be in $\mathscr{C}$, or w_3 would only be hired by f_1 and thus worse off. But f_2 in $\mathscr{C}$ implies that w_1 must be in $\mathscr{C}$. Then f_3 must be in $\mathscr{C}$, so w_2 must also be in $\mathscr{C}$—a contradiction, as w_2 is worse off.

But is $\hat{\mu}$ a reasonable prediction? Under $\hat{\mu}$, f_1 is matched to w_2 and w_3, but would in fact prefer to fire w_2. The problem is that f_1 is not "allowed" to fire w_2 because—as argued above—f_1 would have to form a block that includes w_2 and w_2 is worse off if she is fired. $\Diamond$

Example 2.1 shows that core matchings need not be "individually rational." There are actions, like firing a worker, that an agent should be able to implement on its own, but that the definition of core ends up tying into a larger coalition. In the example, the members of the larger coalition include the worker to be fired, so not all members of the coalition will be better off by the firing.

An additional problem, pointed out by Blair (1988) and Roth and Sotomayor (1990, page 177) is that core matchings may not be pairwise stable (see also Sotomayor 1999).

2.2 *Many-to-many vs. many-to-one*

There is a large literature on one-to-one and many-to-one matchings. Many-to-many matchings constitute a more general model. We have argued that, in many cases, the generality matters. Here we present an example supporting our argument; we observe that the presence of a few many-to-many contracts can change the matching outcome for all agents.

In Example 2.2, if *one* worker is allowed to match with more than one firm, the resulting stable/core matching changes for a large number of agents. Thus, even in markets where one-to-one, or many-to-one, is the rule, a few many-to-many contracts can make a big difference. The example is important because most labor markets have at least a few many-to-many contracts. Thus one needs a many-to-many model to study labor markets.

EXAMPLE 2.2. Let $W=\{\overline{w},w_1,\dots,w_{2K}\}$ and $F=\{f_1,\dots,f_K,\overline{f}\}$. The preferences of workers w_k, for $k=1,\dots,2K$, are the same:

$$P(w_k): f_1, f_2, \dots, f_K, \overline{f}.$$

The preferences of $\overline{w}$ are

$$P(\overline{w}): f_1\overline{f}, \overline{f}, f_1.$$

The preferences of firms f_k for $k=2,\dots,K$ are

$$P(f_k): w_{2k-2}w_{2k-1}, w_{2k-1}w_{2k}, \overline{w}w_{2k}.$$

Firms f_1 and $\overline{f}$ have preferences

$$\begin{aligned} P(f_1):&\quad \overline{w}w_1, w_1w_2 \\ P(\overline{f}):&\quad \overline{w}, w_1, w_2, \dots, w_K. \end{aligned}$$

Consider matchings μ and μ' defined by

$$\begin{array}{lcccccccc} & f_1 & f_2 & \dots & f_k & \dots & f_K & \overline{f} \\ \mu = & w_1w_2 & w_3w_4 & \dots & w_{2k-1}w_{2k} & \dots & w_{2K-1}w_{2K} & \overline{w} \\ \mu' = & \overline{w}w_1 & w_2w_3 & \dots & w_{2k-2}w_{2k-1} & \dots & w_{2K-2}w_{2K-1} & \overline{w}. \end{array}$$

First, if $\overline{w}$ is not allowed to match with more than one firm, then μ is the unique core (and stable) matching. If $\overline{w}$ is allowed to match with more than one firm, then $\langle\{\overline{w}\},\{f_1,\overline{f}\},\mu'\rangle$ blocks μ. Further, μ' is the unique core matching. ◊

The story behind the example should be familiar to academics. Suppose that firms are universities. All workers w_k agree about the ranking of firms: f_1 is the best, followed by f_2, etc. Firm $\overline{f}$ is the worst. However, worker $\overline{w}$, an established and coveted researcher in her field, has a strong desire to work at $\overline{f}$ for geographic reasons ($\overline{f}$ is in the town where $\overline{w}$ grew up, and that is where her family lives). If part-time (many-to-many) appointments are not allowed, $\overline{w}$ will work only for $\overline{f}$ and the resulting matching is, in all likelihood, μ. On the other hand, if $\overline{w}$ is allowed to have part-time appointments at $\overline{f}$ and f_1, μ' results.

3. Preliminary definitions

A (strict) *preference relation* P on a set X is a complete, anti-symmetric, and transitive binary relation on X. We denote by R the weak preference relation associated to P; so $x\ R\ y$ if and only if $x=y$ or $x\ P\ y$. If A is a set, we refer to a collection of preference relations $(P(a))_{a\in A}$ as a *preference profile*.

3.1 *The model*

The model has three primitive components:

- a finite set W of workers
- a finite set F, disjoint from W, of firms
- a preference profile $P=(P(a))_{a\in F\cup W}$, where $P(a)$ is a preference relation over 2^F if $a\in W$ and over 2^W if $a\in F$.

For any agent $a \in F \cup W$, a *set of partners of* a is a subset of F if $a \in W$ and a subset of W if $a \in F$.

Denote the preference profile $(P(w))_{w \in W}$ by $P(W)$ and $(P(f))_{f \in F}$ by $P(F)$.

The assignment problem consists of matching workers with firms, allowing that some firms or workers remain unmatched.

Formally, a *matching* μ is a mapping from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that for all $w \in W$ and $f \in F$:

- $\mu(w) \in 2^F$
- $\mu(f) \in 2^W$
- $f \in \mu(w)$ if and only if $w \in \mu(f)$.

We denote by $\mathscr{M}$ the set of all matchings.

Given a preference relation $P(a)$, the sets of partners preferred by a to the empty set are called *acceptable*.

Given an agent a and a set of partners S of a, let $Ch(S, P(a))$ denote agent a's most-preferred subset of S according to a's preference relation $P(a)$. So $Ch(S, P(a))$ is the unique subset S' of S such that $S' \, P(a) \, S''$ for all $S'' \subseteq S$, $S'' \neq S'$.

We should note that we believe our main results extend to the model of matching with contracts proposed by Hatfield and Milgrom (2005). In their model, each matching specifies not only how individuals are matched, but also additional characteristics of how they match (for example, at which salary). Our method of studying the fixed points of a certain operator is similar to theirs, and our results seem to translate to their model.

We do not explicitly impose bounds on the number of partners an agent can have, called "quotas" in the literature. These can be incorporated in the agents' preferences by requiring that sets are partners that are too large be unacceptable.

3.2 *Individual rationality, stability, and core*

For each agent a, let $P(a)$ be a preference relation. A matching μ is *individually rational* if $\mu(a) \, R(a) \, A$, for all $A \subseteq \mu(a)$, for all $a \in W \cup F$. Hence a matching is individually rational if and only if

$$\mu(a) = Ch(\mu(a), P(a)),$$

for all $a \in F \cup W$.

Individual rationality captures the idea that links are voluntary: if agent a prefers a proper subset $A \subsetneq \mu(a)$ of partners over $\mu(a)$, then she will upset μ by severing her links to the agents in $\mu(a) \setminus A$. The definition is from Roth and Sotomayor (1990, p. 173).

Let $w \in W$, $f \in F$, and let μ be a matching. The pair (w, f) is a *pairwise block* of μ if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup \{w\}, P(f))$, and $f \in Ch(\mu(w) \cup \{f\}, P(w))$.

Definition 3.1. A matching μ is *pairwise stable* if it is individually rational and there is no pairwise block of μ. Denote the set of pair-wise stable matchings by $S(P)$.

DEFINITION 3.2. A *block* of a matching μ is a triple $\langle W', F', \mu' \rangle$, where $F' \subseteq F$, $W' \subseteq W$, and $\mu' \in \mathscr{M}$ are such that

(i) $F' \cup W' \neq \emptyset$

(ii) $\mu'(s) \subseteq F' \cup W'$ for all $s \in F' \cup W'$

(iii) $\mu'(s)\, R(s)\, \mu(s)$ for all $s \in F' \cup W'$

(iv) $\mu'(s)\, P(s)\, \mu(s)$ for some $s \in F' \cup W'$.

In words, a block of a matching μ is a "recontracting" between a subset of workers and firms, so that the agents who recontract are all weakly better off, and at least one of them is strictly better off. Say that $\langle W', F', \mu' \rangle$ *blocks* μ if $\langle W', F', \mu' \rangle$ is a block of μ.

DEFINITION 3.3. A matching μ is a *core matching* if there are no blocks of μ. Denote the set of core matchings by $C(P)$.

An example with an empty core is Example 2 in Konishi and Ünver (2005). We do not reproduce the example here, but one can verify that it falls under the strongest of our restrictions on preferences. Thus the setwise-stable set, for example, is nonempty when the core may be empty.

3.3 *Substitutability*

DEFINITION 3.4. An agent a's preference relation $P(a)$ satisfies *substitutability* if, for any sets S and S' of partners of a with $S \subseteq S'$,

$$b \in Ch(S' \cup b, P(a)) \text{ implies } b \in Ch(S \cup b, P(a)).$$

A preference profile $P = (P(a))_{a \in A}$ is *substitutable* if, for each agent $a \in A$, $P(a)$ satisfies substitutability.

4. SETWISE STABILITY

4.1 *The setwise-stable set*

DEFINITION 4.1. A *setwise block* to a matching μ is a triple $\langle W', F', \mu' \rangle$, where $F' \subseteq F$, $W' \subseteq W$, and $\mu' \in \mathscr{M}$ are such that

(i) $F' \cup W' \neq \emptyset$

(ii) $\mu'(s) \setminus \mu(s) \subseteq F' \cup W'$ for all $s \in F' \cup W'$

(iii) $\mu'(s)\, P(s)\, \mu(s)$ for all $s \in F' \cup W'$

(iv) $\mu'(s) = Ch(\mu'(s), P(s))$ for all $s \in F' \cup W'$.

DEFINITION 4.2. A matching μ is in the *setwise-stable set* if μ is individually rational, and there are no setwise blocks to μ. Denote the set of setwise-stable matchings by $SW(P)$.

This definition of $SW(P)$ is from Sotomayor (1999, Definition 2, pages 59–60). The crucial difference between setwise stability and the core is in item (ii) of Definitions 3.2 and 4.1. By item (ii) of Definition 4.1, a setwise block needs only involve the agents who form *new* matches in the block. The justification is that one only needs an agent's consent to form a new match, not to maintain an existing match. Definition 3.2, in contrast, requires all agents involved in the blocking match to be part of the block; this feature is why the attempted "firing" fails Example 2.1.

Item (iv) of Definition 4.1 contains an element of forward-looking behavior: we consider only blocks that are individually rational, so the agents who engage in blocking do not have an incentive to defect from the block. Item (4) suggests a relation with some notion of a bargaining set, a forward looking solution. We prove in Section 6 that a relation exists.

Recall Example 2.1. We argued that the core matching $\hat{\mu}$ (in fact the unique core matching) is not a good prediction. Consider, instead, the matching defined by $\mu(w_i) = \{f_i\}$, for $i = 1, 2, 3$. It is easy, if somewhat cumbersome, to check that μ is setwise stable.

It also has some interest to see why μ in Example 2.1 is not a core matching; $\langle W, F, \hat{\mu}\rangle$ blocks μ, as $\hat{\mu}(w_1) = \{f_2, f_3\}\ P(w_1)\ \mu(w_1)$, $\hat{\mu}(w_2) = \{f_1, f_3\}\ P(w_2)\ \mu(w_2)$, and $\hat{\mu}(w_3) = \{f_1, f_2\}\ P(w_3)\,\mu(w_3)$. Similarly for firms. But this is a block from which all agents wish to unilaterally deviate. We characterize the blocks of setwise-stable matchings in Section 8.

Note, incidentally, that μ is not Pareto optimal, as it is Pareto dominated by $\hat{\mu}$. Thus setwise stable matchings need not be Pareto optimal.

4.2 *The individually-rational core*

Definition 4.3. A block $\langle W', F', \mu'\rangle$ is *individually rational* if $\mu'(s) = Ch(\mu'(s), P(s))$ for all $s \in F' \cup W'$.

Definition 4.4. A matching μ is in the *individually-rational core* if it is individually rational and has no individually-rational blocks. Denote the set of individually-rational core matchings by $IRC(P)$.

Sotomayor (1999) restricts attention to individually-rational matchings. So she implicitly refers to the individually-rational core.

4.3 *A bargaining set*

Let μ be a matching.

Definition 4.5. An *objection* to μ is a triple $\langle W', F', \mu'\rangle$, where $F' \subseteq F$, $W' \subseteq W$, and $\mu' \in \mathscr{M}$ are such that (i), (ii), and (iii) from Definition 4.1 are satisfied.

Let $\langle W', F', \mu'\rangle$ be an objection to μ. A *counterobjection* to μ is an objection $\langle W'', F'', \mu''\rangle$ to μ' such that $F'' \subseteq F'$ and $W'' \subseteq W'$.

Definition 4.6. A matching μ is in the *bargaining set* if μ is individually rational and there are no objections without counterobjections to μ. Denote the bargaining set by $B(P)$.

The bargaining set reflects forward-looking agents; an objection would not be implemented if the agents who must carry out the objection have incentives to deviate.

For the one-to-one model, Klijn and Massó (2003) prove that Zhou's bargaining set (Zhou 1994) coincides with a weak pairwise-stability solution. The bargaining set we propose differs from Zhou's because counterobjections are allowed only from "within" the objecting coalition. With more general counterobjections, one gets a larger solution, and our results would then imply that the larger solution is nonempty. But Zhou's bargaining set rules out counterobjections that come only from within—so our results do not imply that Zhou's bargaining set is nonempty. Still, it seems to us that $B(P)$ captures the strategic reasoning underlying Zhou's bargaining set.

4.4 *The Blair core*

We introduce a solution using Blair's (1988) order on sets of partners to define blocks

The definition of a block (Definition 3.2) formally makes sense for any profile of binary relations $(B(s))_{s\in F\cup W}$. Accordingly, one can define the core matchings $C(B)$ for any profile $B=(B(s))_{s\in F\cup W}$ of binary relations.

In particular, given a preference profile $P=(P(s))$, we can construct a binary relation $R^B=(R^B(s))$ by saying that $A\ R^B(s)\ D$ if and only if $A=D$ or $A=Ch(A\cup D,P(s))$. The strict relation P^B is $A\ R^B(s)\ D$ if and only if $A\neq D$ and $A=Ch(A\cup D,P(s))$. We call the resulting core, $C(P^B)$, the *Blair-Core*, as Blair (1988) introduced the relation P^B.

Note that a matching μ is in the Blair-Core if it is immune to deviations μ' such that $\mu'(a)=Ch(\mu'(a)\cup\mu(a),P(a))$. But $\mu'(a)=Ch(\mu'(a)\cup\mu(a),P(a))$ is only sufficient, and not necessary, for $\mu'(a)\ P(a)\ \mu(a)$. So the Blair-Core contains more matchings than the core.

4.5 *Strong pairwise stability*

We introduce a solution that requires stability against blocks by a firm and a set of workers. This solution plays an auxiliary role in our results (similar to its role in Echenique and Oviedo 2004); it reflects the effect of strengthening the structure on only one side of the market.

A pair $(D,f)\in 2^W\times F$ with $D\neq\emptyset$ *blocks** μ if $D\cap\mu(f)=\emptyset$, $D\subseteq Ch(\mu(f)\cup D,P(f))$, and $f\in Ch(\mu(w)\cup f,P(w))$ for all $w\in D$.

DEFINITION 4.7. A matching μ is *stable** if it is individually rational and there is no pair $(D,f)\in 2^W\times F$ that blocks* μ. Denote the set of stable* matchings by $S^*(P)$.

5. A fixed-point approach

We construct a map T on the set of "pre-matchings," a superset of $\mathscr{M}$. We shall use the fixed points of T to prove results about the various notions of stability.

5.1 *Pre-matchings*

Say that a pair $\nu=(\nu_F,\nu_W)$ with $\nu_F:F\to 2^W$ and $\nu_W:W\to 2^F$ is a *pre-matching*. Let $\mathscr{V}_W$ ($\mathscr{V}_F$) denote the set of all ν_W (ν_F) functions. Thus, $\mathscr{V}_F=(2^W)^F$, $\mathscr{V}_W=(2^F)^W$. Denote the set of pre–matchings $\nu = (\nu_F, \nu_W)$ by $\mathscr{V} = \mathscr{V}_F \times \mathscr{V}_W$. We often refer to $\nu_W(w)$ by $\nu(w)$ and to $\nu_F(f)$ by $\nu(f)$.

A pre-matching ν is a matching if ν is such that $\nu_W(w) = f$ if and only if $w \in \nu_F(f)$.

5.2 *The map T*

Let ν be a pre-matching, and let

$$U(f,\nu) = \{w \in W : f \in Ch(\nu(w) \cup \{f\}, P(w))\}$$

and

$$V(w,\nu) = \{f \in F : w \in Ch(\nu(f) \cup \{w\}, P(f))\}.$$

The set $V(w,\nu)$ is the set of firms f that are willing to hire w, possibly after firing some of the workers it was assigned by ν. The set $U(f,\nu)$ is the set of workers w that are willing to add f to its set of firms $\nu(w)$, possibly after firing some firms in $\nu(w)$.

Now, define $T : \mathscr{V} \to \mathscr{V}$ by

$$(T\nu)(s) = \begin{cases} Ch(U(s,\nu), P(s)) & \text{if } s \in F \\ Ch(V(s,\nu), P(s)) & \text{if } s \in W. \end{cases}$$

The map T has a simple interpretation: $(T\nu)(f)$ is firm f's optimal team of workers, among those willing to work for f, and $(T\nu)(w)$ is the set of firms preferred by w, among the firms that are willing to hire w.

Let the *fix-point set* be the set of fixed points of T; we denote it by $\mathscr{E}(P)$. Formally,

$$\mathscr{E}(P) = \{\nu \in \mathscr{V} : \nu = T\nu\}.$$

Recall Example 2.1 and matching μ from Section 4.1. Note that μ is a setwise-stable matching and a fixed-point of T: $V(w_1,\mu) = \{f_1, f_2\}$, so $\{f_1\} = Ch(V(w_1,\mu), P(w_1))$, $V(w_2,\mu) = \{f_2, f_3\}$, so $\{f_2\} = Ch(V(w_2,\mu), P(w_2))$, and $V(w_3,\mu) = \{f_1, f_3\}$, so $\{f_3\} = Ch(V(w_3,\mu), P(w_2))$. Similarly for firms.

Further, $\hat{\mu}$, the core matching in Example 2.1, is not a fixed-point of T, as $U(f_1,\hat{\mu}) = \{w_2\}$ and $\{w_2, w_3\} \neq Ch(U(f_1,\hat{\mu}), P(f_1))$.

We now describe an algorithm that is associated with the techniques we use to prove our results: the techniques exploit the fixed points of T, and the algorithm is designed to find a fixed point of T. The definition of the algorithm is very simple.

DEFINITION 5.1. The *T-algorithm* is the procedure of iterating T, starting at some pre-matching ν.

Say that the T-algorithm *stops* at $\nu' \in \mathscr{V}$ if there is $\nu \in \mathscr{V}$ and K such that $\nu' = T^k(\nu)$ for all $k \geq K$. Note that the T-algorithm stops at ν' only if $\nu' \in \mathscr{E}(P)$.

The prematching at which one starts the iterations of T matters. We consider two candidates: Let ν_0 and ν_1 be the prematchings defined by $\nu_0(f) = \nu_1(w) = \emptyset$, $\nu_0(w) = F$, and $\nu_1(f) = W$ for all w and f. We consider the T-algorithm starting at prematchings ν_0 and ν_1.

The T-algorithm is different from Gale and Shapley's (1962) deferred-acceptance algorithm, in the sense that it behaves differently in certain cases. Echenique and Oviedo (2004) present an example where Gale and Shapley's algorithm is "fooled" and finds a matching that is not stable; the T-algorithm, in contrast, never finds a non-stable matching. That said, the T-algorithm performs intuitively similar steps to Gale and Shapley's; it "offers" matches sequentially to an agent's best available partners. And, under the right structure on preferences, both algorithms find the same pairwise-stable matching. See Echenique and Oviedo (2004) for a more detailed comparison of the algorithms.[2]

6. Nonemptiness of solutions and relationships among solutions

We organize the results according to the structure needed on preferences. In some results, we impose structure on only one side of the market. We always impose weakly more structure on workers' preferences. The model is symmetric, so it should be clear that appropriate versions of the results are true, interchanging the structure on workers' and firms' preferences.

The following table lists the solutions for easy reference when reading the results.

$S(P)$	Pairwise stable set (Definition 3.1)
$C(P)$	Core (Definition 3.3)
$SW(P)$	Setwise stable set (Definition 4.2)
$IRC(P)$	Individually-rational core (Definition 4.4)
$B(P)$	Bargaining set (Definition 4.6)
$S^*(P)$	Strong pairwise stability (Definition 4.7)
$\mathscr{E}(P)$	Fix-Point set

Table 1 on page 256 contains a summary of results in the paper.

6.1 *Results under substitutability*

Theorem 6.1. $\mathscr{E}(P) \subseteq S^*(P) \subseteq S(P)$. *Further:*

(i) If $P(W)$ is substitutable, then

$$S^*(P) = \mathscr{E}(P) \subseteq C(P^B).$$

(ii) If P is substitutable, then $S(P) = \mathscr{E}(P)$, $\mathscr{E}(P)$ is nonempty, and the T-algorithm stops at a matching in $\mathscr{E}(P)$.

For a proof see Section 11.

[2]The T-algorithm is also different from Martínez et al.'s (2004) algorithm for finding all pairwise-stable matchings; their algorithm builds on Gale and Shapley's algorithm by successively truncating preferences (while maintaining substitutability) to find all the pairwise-stable matchings.

6.2 *Results under strong substitutability*

DEFINITION 6.2. An agent a's preference ordering $P(a)$ satisfies *strong substitutability* if, for any sets S and S', with $S' \, P(a) \, S$,

$$b \in Ch(S' \cup b, P(a)) \text{ implies } b \in Ch(S \cup b, P(a)).$$

Say that a preference profile P is *strongly substitutable* if $P(a)$ satisfies strong substitutability for every agent a.

PROPOSITION 6.3. *If $P(a)$ satisfies strong substitutability, then it satisfies substitutability.*

PROOF. Let S and S' be sets of agents, with $S \subseteq S'$. Suppose that $b \in C' = Ch(S' \cup b, P(a))$. We now prove that $b \in Ch(S \cup b, P(a))$.

Note that $C' = Ch(C', P(a))$. Now, $S \cup b \subseteq S' \cup b$ implies that $C' \, R(a) \, S \cup b$. If $C' = S \cup b$ then $S \cup b = Ch(S \cup b, P(a))$ and we are done. Let $C' \, P(a) \, S \cup b$. Then $b \in Ch(S \cup b, P(a))$, as $P(a)$ satisfies strong substitutability. □

THEOREM 6.4. *$SW(P) \subseteq \mathscr{E}(P)$ and $B(P) \subseteq \mathscr{E}(P)$. Further, if $P(F)$ is substitutable and $P(W)$ is strongly substitutable, then $\mathscr{E}(P) = SW(P) = B(P)$ and $\mathscr{E}(P) \subseteq IRC(P)$.*

For a proof, see Section 12.

Thus, when one side of the market has strongly substitutable preferences, we can characterize the setwise-stable set. In light of Proposition 6.3, Theorem 6.4 implies that $S(P) = SW(P)$ and that $S(P) = B(P)$.

THEOREM 6.5. *If $P(F)$ is substitutable and $P(W)$ is strongly substitutable, then $S(P)$, $IRC(P)$, $SW(P)$, and $B(P)$, are nonempty. The T-algorithm finds a matching in $S(P)$, $IRC(P)$, $SW(P)$, and $B(P)$.*

For a proof, see Section 12.

REMARK 6.6. Example 4 in Konishi and Ünver (2005) has agents with substitutable preferences and a pairwise-stable matching that is not in $SW(P)$, thus Theorem 6.4 is tight. (We thank Hideo Konishi and Utku Ünver for pointing this out.) Sotomayor (1999) presents an example where the set of setwise-stable matchings is empty. One can show that preferences in her example are not strongly substitutable.

REMARK 6.7. We can weaken the definition of strongly substitutable as follows. For all S and S' with $S = Ch(S, P(a))$, $S' = Ch(S', P(a))$, and $S' \, P(a) \, S$,

$$b \in Ch(S' \cup b, P(a)) \text{ implies } b \in Ch(S \cup b, P(a)).$$

All our results go through under this weaker definition. We chose the stronger formulation in our exposition to make the comparison with earlier work easier—it makes comparison with substitutability easier. But when we check that an example violates strong substitutability, we check for the weaker version.

6.3 *Discussion of strong substitutability*

How strong is the assumption of strong substitutability? We lack a characterization of strong substitutability—just as a characterization of traditional (Kelso–Crawford) substitutability is unavailable.[3] But we give a feeling for the assumption by discussing preferences that are built from preferences over individual workers.

First, strong substitutability is weaker than the assumption of separability used in matching models (Crawford and Knoer 1981; Dutta and Massó 1997; Sönmez 1996). Separability says that, for any set of partners S, $S \cup b \; P \; S \setminus b$ if and only if $b \; P \; \emptyset$ (separability has been used quite extensively in social choice theory; e.g. Barberà et al. (1991)). The proof that separability implies strong substitutability is straightforward; we omit it.

The statement that separability implies strong substitutability is true when agents do not have quotas; a quota for an agent a is a number q such that if S has more than q members, then $\emptyset \; P \; S$. Thus it applies for example to Kelso and Crawford's (1982) original model, as well as the treatment in Chapter 6 of Roth and Sotomayor (1990). These models rule out that players have quotas.

Second, strong substitutability is not stronger than responsiveness, another common assumption is the matching literature (see Roth and Sotomayor (1990) for a definition of responsiveness). One can easily write examples of non-responsive preferences that satisfy strong substitutability.

Third, to give a feeling for how restrictive strong substitutability is, consider the following example with four workers and a quota of two.[4]

EXAMPLE 6.8. Let $W = \{w_1, w_2, w_3, w_4\}$. Suppose that a firm has preferences over individual workers $w_1 \; P \; w_2$, $w_2 \; P \; w_3$ and $w_3 \; P \; w_4$.

Suppose the firm has a quota of 2, so only sets with two or fewer elements are acceptable. How can we rank the sets

$$\{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_2, w_4\}, \{w_3, w_4\}$$

building from preferences over individuals? Obviously we need $\{w_1, w_2\} \; P \; \{w_1, w_3\}$, $\{w_2, w_3\} \; P \; \{w_3, w_4\}$, and so on. There are two possibilities:

$$P_1 : w_1 w_2, w_1 w_3, \boxed{w_1 w_4, w_2 w_3}, w_2 w_4, w_3 w_4, w_1, w_2, w_3, w_4$$

$$P_2 : w_1 w_2, w_1 w_3, \boxed{w_2 w_3, w_1 w_4}, w_2 w_4, w_3 w_4, w_1, w_2, w_3, w_4$$

The ranking of $\{w_1, w_4\}$ and $\{w_2, w_3\}$ is undetermined; P_1 ranks $\{w_1, w_4\}$ first, P_2 ranks $\{w_2, w_3\}$ first. Both P_1 and P_2 are substitutable, but only P_2 is strongly substitutable: Note that $w_4 \in Ch(\{w_1 w_4\}, P_1)$ and $\{w_1, w_4\} \; P_1 \; \{w_2, w_3\}$, but $w_4 \notin Ch(\{w_2, w_3\} \cup w_4, P_1)$. So P_1 is not strongly substitutable. It is simple, if tedious, to check that P_2 is strongly substitutable. ◊

[3] Echenique (2004) has a partial characterization that allows him to count the number of substitutable rules and show they are a small set of rules.

[4] The right separability assumption for models with quotas is q-separability, developed by Martínez et al. (2000). But q-separability does not imply strong substitutability.

Example 6.8 points to a general procedure for obtaining strongly-substitutable preferences from preferences over individuals when there is a quota. Let $(S \cup b)\ P\ (S \setminus b)$ if and only if $b\ P\ \emptyset$, unless S has the maximum number of elements allowed by the quota. If S and S' have the maximum number of elements, let $S\ P\ S'$ if the worst agent in S is preferred to the worst agent in S'.

Example 6.8 suggests that strong substitutability may be a strong assumption when a) agents have a quota, b) agents have responsive preferences, and c) all sets that exhaust the quota are acceptable. Even in this case, we believe strong substitutability is valuable because it provides the wealth of results we document here and is easily interpretable economically.

Finally, in applications, the set of acceptable partners is often quite small. And both substitutability and strong substitutability are less restrictive if fewer sets of partners are acceptable (Remark 6.7). For example, the size of the set of acceptable hospitals in the National Resident Matching Program in 2003 was on average 7.45, out of 3719 programs.[5] To be fair, this number could be artificially low because agents strategically report shorter rank-order lists in the NRMP. In a recent proposal to match high schools and students in New York City by a Gale-Shapley algorithm, students would be required to rank 12—out of over 200—acceptable high schools. Students and parents complain that 12 is too long a list (New York Times story by David M. Herszenhorn, "Revised Admission for High Schools," on October 3rd, 2003).

When the set of acceptable partners is small, strong substitutability is quite common, at least in the sense that one struggles to find preferences that do not satisfy it. See, for example, Examples 6.6 in Roth and Sotomayor (1990) and 5.2 in Blair (1988).

7. An implementation of $SW(P)$

We present a simple (non-cooperative) bargaining game. The set of subgame-perfect Nash equilibrium (SPNE) outcomes of the game coincides with the setwise-stable set, so the game fully implements $SW(P)$ in a complete-information environment.

The game is described as follows. First, every firm f proposes a set of partners $\eta_f \subseteq W$. Firms make these proposals simultaneously. Second, after observing all the firms' proposals, each worker w proposes a set of partners $\xi_w \subseteq F$. Workers make these proposals simultaneously. Finally, a matching μ results by $w \in \mu(f)$ if and only if $w \in \eta_f$ and $f \in \xi_w$. In words, w and f are matched if and only if f proposes w as its partner and w proposes f as its partner. Given a preference profile P, this description defines an extensive-form game, $\Gamma(P)$.

A strategy for a firm f is a proposal $\eta_f \subseteq W$. A strategy for a worker w is a collection, ξ_w, with one proposal $\xi_w(\eta) \subseteq F$ for each profile $\eta = (\eta_f)_{f \in F}$ of firms' proposals. A strategy profile (η^*, ξ^*) is a *subgame-perfect Nash equilibrium* (SPNE) of $\Gamma(P)$ if, for all w and f,

$$\xi^*_w(\eta) \cap \{\tilde{f} : w \in \eta_{\tilde{f}}\}\ R(w)\ A,$$

[5] Source: http://www.nrmp.org/, accessed in 2004.

for all $A \subseteq \{\tilde{f} : w \in \eta_{\tilde{f}}\}$ and

$$\eta^*_f \cap \{w : f \in \xi^*_f(\eta)\}\ R(f)\ A \cap \{w : f \in \xi^*_f(A, \eta^*_{-f})\}$$

for all $A \subseteq W$. In words, (η^*, ξ^*) is an SPNE if $\xi^*_w(\eta)$ is an optimal proposal given the firms' proposal η, and η^*_f is optimal given the other firms' proposals η^*_{-f} and the workers' proposals.

THEOREM 7.1. *Let $P(W)$ be substitutable. A matching μ is the outcome of a subgame-perfect Nash equilibrium of $\Gamma(P)$ if and only if $\mu \in \mathscr{E}(P)$.*

See Section 13 for a proof. Theorems 6.4 and 7.1 imply the following result.

COROLLARY 7.2. *Let $P(F)$ be substitutable and $P(W)$ be strongly substitutable. A matching μ is the outcome of a subgame-perfect Nash equilibrium if and only if $\mu \in SW(P)$.*

The implication of Theorem 7.1 and Corollary 7.2 is that the setwise-stable matchings are exactly those consistent with a basic non-cooperative bargaining model. Thus core matchings, for example, are not guaranteed to be SPNE outcomes.

8. Blocks of setwise-stable matchings

In Example 2.1, $\hat{\mu}$ blocks μ through a coordinated and non-individually-rational effort of all agents. The preferences in Example 2.1 exhibit agents a who want agents b, where b dislikes a but is willing to accept a if she gets c, who dislikes b, and so on until a cycle is closed. We call such a cycle an acceptance-rejection cycle.

We now show that a matching in $\mathscr{E}(P)$ can, in fact, be blocked only through an effort of this kind.

DEFINITION 8.1. Let μ be a matching. An agent a wants to *add* an agent b to her partners if

$$b \in Ch(\mu(a) \cup b, P(a)).$$

An alternating sequence of workers and firms

$$(w_0, f_0, w_1, f_1, \ldots, w_K, f_K),$$

with $(w_0, f_0) = (w_K, f_K)$, is an *acceptance-rejection cycle for* μ if, for k with $0 \le k \le K-1$, w_k wants to add f_k to her partners but f_k does not want to add w_k, while f_k wants to add w_{k+1} to her partners and w_{k+1} does not want to add f_k.

THEOREM 8.2. *Let P be substitutable. If $\mu \in \mathscr{E}(P)$ and $\langle W', F', \mu' \rangle$ is a block of μ, then there is an acceptance-rejection cycle for μ in $\mu'(F' \cup W') \setminus \mu(F' \cup W')$.*

See Section 13 for a proof. Theorems 6.4 and 8.2 imply

COROLLARY 8.3. *Let $P(F)$ be substitutable and $P(W)$ strongly substitutable. If $\mu \in SW(P)$ and $\langle W', F', \mu' \rangle$ is a block of μ, then there is an acceptance-rejection cycle for μ in $\mu'(F' \cup W') \setminus \mu(F' \cup W')$.*

9. Lattice structure

We now present results on the lattice structure of our solution concepts. These results have an interpretation in terms of the opposition and coincidence of agents' preferences for the various matchings in the solution: opposition because what is good for one side of the market is bad for the other, and coincidence because there are certain matchings that all agents on one side of the market find the most desirable. We first present our results and then give an interpretation.

9.1 *Preliminary definitions*

Let X be a set and B a partial order on X—a transitive, reflexive, and antisymmetric binary relation. Let $A \subseteq X$. Denote by $\inf_B A$ the greatest lower bound and by $\sup_B A$ the lowest upper bound on A in the order B. Say that the pair $\langle X, B \rangle$ is a *lattice* if, whenever $x, y \in X$, both $x \wedge_B y = \inf_B \{x, y\}$ and $x \vee_B y = \sup_B \{x, y\}$ exist in X. A subset $A \subseteq X$ is a *sublattice* of $\langle X, B \rangle$ if, whenever $x, y \in A$, both $x \wedge_B y \in A$ and $x \vee_B y \in A$.

A lattice $\langle X, B \rangle$ is *distributive* if, for all $x, y, z \in X$, $x \vee_B (y \wedge_B z) = (x \vee_B y) \wedge_B (x \vee_B z)$. Let $\langle X, B \rangle$ and $\langle Y, R \rangle$ be lattices. A map $\psi : X \to Y$ is a *lattice homomorphism* if, for all $x, y \in X$, $\psi(x \wedge_B y) = \psi(x) \wedge_R \psi(y)$ and $\psi(x \vee_B y) = \psi(x) \vee_R \psi(y)$. A map ψ is a *lattice isomorphism* if it is a bijection and a lattice homomorphism.

REMARK 9.1. The product of lattices, when endowed with the product order, is a lattice (Topkis 1998, page 13). The lattice operations are the product of the component lattice operations.

9.2 *Partial orders*

We introduce two partial orders on $\mathscr{V}$. Both orders are such that a prematching ν is smaller than a prematching ν' if all the firms are better off in ν' than in ν and all the workers are better off in ν than in ν'. In the first order, "better off" means that $\nu'(f)$ is the optimal set of workers for f, out of $\nu(f) \cup \nu'(f)$, and similarly for workers. In the second order, "better off" means simply that $\nu'(f)\, P(f)\, \nu(f)$, and similarly for workers.

The first partial order was introduced by Blair (1988) to show that $S(P)$ has a lattice structure under substitutable preferences. The second order is the standard one from one-to-one theory; Blair showed that one does not obtain a lattice structure using this order (even with substitutable preferences).

DEFINITION 9.2. Define the following partial orders on $\mathscr{V}_F$, $\mathscr{V}_W$ and $\mathscr{V}$.

(i) $<^B_F$ on $\mathscr{V}_F$ by $\nu'_F <^B_F \nu_F$ if and only if $\nu'_F \neq \nu_F$ and, for all f in F, $\nu_F(f) = \nu'_F(f)$ or

$$\nu_F(f) = Ch(\nu_F(f) \cup \nu'_F(f), P(f)).$$

(ii) $<^B_W$ on $\mathscr{V}_W$ by $v'_W <^B_W v_W$ if and only if $v'_W \neq v_W$ and, for all w in W, $v_W(w) = v'_W(w)$ or

$$v_W(f) = Ch(v_W(w) \cup v'_W(w), P(w)).$$

(iii) The weak partial orders associated with $<^B_F$ and $<^B_W$ are denoted $\leq^B_F$ and $\leq^B_W$, and defined as: $v'_F \leq^B_F v_F$ if $v_F = v'_F$ or $v'_F <^B_F v_F$, and $v'_W \leq^B_W v_W$ if $v_W = v'_W$ or $v'_W <^B_W v_W$.

(iv) $\leq^B_F$ on $\mathscr{V}$ by $v' \leq^B_F v$ if and only if $v_W \leq^B_W v'_W$ and $v'_F \leq^B_F v_F$. The strict version of $\leq^B_F$ on $\mathscr{V}$ is $v' <^B v$ if $v' \leq^B v$ and $v' \neq v$.

(v) $\leq^B_W$ on $\mathscr{V}$ by $v' \leq^B_W v$ if and only if $v \leq^B_F v'$.

Definition 9.3. Define the following partial orders on $\mathscr{V}_F$, $\mathscr{V}_W$ and $\mathscr{V}$:

(i) $\leq_F$ on $\mathscr{V}_F$ by $v'_F \leq_F v_F$ if $v_F(f)\, R(f)\, v'_F(f)$ for all $f \in F$. The strict version of $\leq_F$ on $\mathscr{V}_F$ is $v'_F <_F v_F$ if $v'_F \leq_F v_F$ and $v'_F \neq v_F$.

(ii) $\leq_W$ on $\mathscr{V}_W$ by $v'_W \leq_W v_W$ if $v_W(w)\, R(w)\, v'_W(w)$ for all $w \in W$. The strict version of $\leq_W$ on $\mathscr{V}_W$ is $v'_W <_W v_W$ if $v'_W \leq_W v_W$ and $v'_W \neq v_W$.

(iii) $\leq_F$ on $\mathscr{V}$ by $v' \leq_F v$ if and only if $v_W \leq_W v'_W$ and $v'_F \leq_F v_F$. The strict version of $\leq_F$ on $\mathscr{V}$ is $v' <_F v$ if $v' \leq_F v$ and $v' \neq v$.

(iv) $\leq_W$ on $\mathscr{V}$ by $v' \leq_W v$ if and only if $v \leq_F v'$.

Definitions 9.2 and 9.3 abuse notation in using each symbol ($\leq^B_F$, $\leq_F$, $\leq^B_W$, and $\leq_W$) for two different orders. The abuse of notation is not, we believe, confusing.

To simplify the notation in the sequel, let $(\leq^B, \leq) \in \{(\leq^B_F, \leq_F), (\leq^B_W, \leq_W)\}$. All statements that follow are true both with $(\leq^B, \leq) = (\leq^B_F, \leq_F)$ and $(\leq^B, \leq) = (\leq^B_W, \leq_W)$.

Remark 9.4. $\leq^B$ is coarser than $\leq$, as $v' \leq^B v$ implies that $v' \leq v$.

Remark 9.5. $\langle \mathscr{V}, \leq_F \rangle$ is a lattice (see Remark 9.1), and the lattice operations are

$$v \vee_F v'(f) = \begin{cases} v(f) & \text{if } v(f)\, R(f)\, v'(f) \\ v'(f) & \text{if } v'(f)\, P(f)\, v(f) \end{cases}$$

and

$$v \vee_F v'(w) = \begin{cases} v'(w) & \text{if } v(w)\, R(w)\, v'(w) \\ v(w) & \text{if } v'(w)\, P(w)\, v(w). \end{cases}$$

The operation $v \wedge_F v'$ is defined symmetrically, giving f the worst of $v(f)$ and $v'(f)$, and giving w the best of $v(w)$ and $v'(w)$.

The pair $\langle \mathscr{V}, \leq_W \rangle$ is a lattice and the lattice operations are analogous to $\vee_F$ and $\wedge_F$.

Blair's order incorporates strong substitutability:

PROPOSITION 9.6. *If $P(a)$ is substitutable, then $P^B(a)$ is strongly substitutable.*

PROOF. Let $b \in Ch(S' \cup b, P(a))$ and $S' \; P^B(a) \; S$. Note that

$$\begin{aligned} b \in Ch(S' \cup b, P(a)) &= Ch(Ch(S \cup S', P(a)) \cup b, P(a)) \\ &= Ch(S \cup S' \cup b, P(f)), \end{aligned}$$

where the first equality is by definition of P^B and the second equality is a property choice rules. Finally, $b \in Ch(S \cup S' \cup b, P(a))$ and substitutability imply that $b \in Ch(S' \cup b, P(a))$. □

Proposition 9.6 explains Blair's results in the light of our results.

9.3 *Lattice structure*

With substitutable preferences, T is an increasing map under order $\leq^B$. Tarski's fixed point theorem then delivers a lattice structure on $\mathscr{E}(P)$. With strongly substitutable preferences, T is an increasing map under order $\leq$. Tarski's fixed point theorem gives a lattice structure on $\mathscr{E}(P)$ under order $\leq$. We discuss the implications below.

THEOREM 9.7. *Let P be substitutable. Then*

(i) $\langle \mathscr{E}(P), \leq^B \rangle$ is a nonempty lattice;

(ii) the T-algorithm starting at v_0 stops at $\inf_{\leq^B_F} \mathscr{E}(P)$ and the T-algorithm starting at v_1 stops at $\sup_{\leq^B_F} \mathscr{E}(P)$.

Further, if P is strongly substitutable, $\langle \mathscr{E}(P), \leq \rangle$ is a nonempty lattice, $\inf_{\leq^B_F} \mathscr{E}(P) = \inf_{\leq_F} \mathscr{E}(P)$, and $\sup_{\leq^B_F} \mathscr{E}(P) = \sup_{\leq_F} \mathscr{E}(P)$.

See Section 14 for a proof.

THEOREM 9.8. *Let $P(F)$ be substitutable and $P(W)$ be strongly substitutable. Then*

(i) if $v, v' \in \mathscr{E}(P)$ are such that $v'(w) \; R(w) \; v(w)$ for all $w \in W$, then $v(f) \; R(f) \; v'(f)$ for all $f \in F$.

(ii) Further, let $P(F)$ be strongly substitutable. If $v, v' \in \mathscr{E}(P)$ are such that $v'(f) \; R(f) \; v(f)$ for all $f \in F$, then $v(w) \; R(w) \; v'(w)$ for all $w \in W$.

See Section 14 for a proof.

By definition of $\leq^B_F$, $\leq_F$, $\leq^B_W$, and $\leq_W$, we get $\inf_{\leq_F} \mathscr{E}(P) = \sup_{\leq_W} \mathscr{E}(P)$, $\inf_{\leq_W} \mathscr{E}(P) = \sup_{\leq_F} \mathscr{E}(P)$, $\inf_{\leq^B_F} \mathscr{E}(P) = \sup_{\leq^B_W} \mathscr{E}(P)$, and $\inf_{\leq^B_W} \mathscr{E}(P) = \sup_{\leq^B_F} \mathscr{E}(P)$.

Theorem 9.7 implies Theorem 6.5. It also implies

COROLLARY 9.9. *If $P(F)$ is substitutable and $P(W)$ is strongly substitutable, then $\langle SW(P), \leq^B\rangle = \langle B(P), \leq^B\rangle = \langle S(P), \leq^B\rangle$ are nonempty lattices. Further, if $P(F)$ is strongly substitutable, $\langle SW(P), \leq\rangle = \langle B(P), \leq\rangle = \langle S(P), \leq\rangle$ are nonempty lattices.*

Theorems 9.7 and 9.8 have an interpretation in terms of worker-firm "conflict" and worker-worker (or firm-firm) "coincidence" of interests (Roth 1985).

First, Theorem 9.7 implies that there are two distinguished matchings in $\mathscr{E}(P)$. One is simultaneously better for all firms, and worse for all workers, than any other matching in $\mathscr{E}(P)$. The other is simultaneously worse for all firms, and better for all workers, than any other matching in $\mathscr{E}(P)$. The lattice structure thus implies a coincidence-of-interest property.

Second, Theorem 9.8 reflects a global worker-firm conflict of interest over $\mathscr{E}(P)$; for any two matchings in $\mathscr{E}(P)$, if one is better for all firms it must also be worse for all workers, and vice versa. Roth (1985) proved that Statement i in Theorem 9.8 holds in the one-to-one model and in the many-to-one model. Roth also proved that Statement ii in Theorem 9.8 holds in the one-to-one model. Here we extend Roth's results, as workers' preferences are trivially strongly substitutable in the many-to-one model, and all agents' preferences are trivially strongly substitutable in the one-to-one model.[6]

In light of Theorem 6.1, Theorem 9.9 implies that $\langle S(P), \leq_F^B\rangle$ is a lattice when preferences are substitutable—a result first proved by Blair (1988). Blair shows with an example that $\langle S(P), \leq_F\rangle$ may not be a lattice. Preferences in Blair's example are not strongly substitutable; we discuss Blair's example in Section 9.5.

In the one-to-one model, the lattice-structure of $\langle S(P), \leq_F\rangle$ has been known since at least Knuth (1976) (Knuth attributes the result to J. Conway). Theorem 9.7 extends the result to the many-to-many model, as preferences are trivially strongly substitutable in the one-to-one model.

9.4 *Further conflict/coincidence properties*

Two additional features of many-to-one and one-to-one matchings merit attention.

9.4.1 *Stronger coincidence-of-interest property* Roth (1985) presents a stronger version of the coincidence-of-interest property implicit in the result that $\langle \mathscr{E}(P), \leq_F^B\rangle$ is a lattice. He proves that if μ and μ' are pairwise-stable matchings in the many-to-one model, the matching that gives each firm f its best subset out of $\mu(f) \cup \mu'(f)$ is stable and worse than both μ and μ' for all workers.

Roth's stronger coincidence-of-interest property does not extend to the many-to-many model with strongly substitutable preferences. Example 5.2 in Blair (1988) is a counterexample—we discuss this example in Section 9.5.

But note

PROPOSITION 9.10. *Let $P(F)$ be substitutable. Let $\mu, \mu' \in S(P)$. Define the matching $\hat{\mu}$ by $\hat{\mu}(f) = Ch(\mu(f) \cup \mu'(f), P(f))$, for all $f \in F$. If $\hat{\mu}(w) \in \{\mu(w), \mu'(w)\}$ for all $w \in W$, then*

[6] By Proposition 9.6, we also extend Blair's (1988) version of Roth's result (Blair's Lemmas 4.3 and 4.4).

$\hat{\mu} \in S(P)$. *Further, if* $P(W)$ *is substitutable, then* $\mu(w)\, R(w)\, \hat{\mu}(w)$ *and* $\mu'(w)\, R(w)\, \hat{\mu}(w)$ *for all* $w \in W$.

PROOF. The proof that $\hat{\mu} \in S(P)$ is a minor variation of Roth's (1985) proof of the coincidence-of-interest property in the many-to-one model.

First, $\hat{\mu}$ is individually rational: both μ and μ' are individually rational, so $\hat{\mu}(w) = Ch(\hat{\mu}(w), P(w))$ for all w; by definition of $\hat{\mu}$, $\hat{\mu}(f) = Ch(\hat{\mu}(f), P(f))$ for all f.

Second, there are no pairwise blocks of $\hat{\mu}$. Suppose (w, f) is a pairwise block of $\hat{\mu}$. Then $w \notin \hat{\mu}(f)$, $f \in Ch(\hat{\mu}(w) \cup f, P(w))$, and $w \in Ch(\hat{\mu}(f) \cup w, P(f))$. Without loss of generality, let $\hat{\mu}(w) = \mu(w)$. So $f \notin \mu(w)$ and $f \in Ch(\mu(w) \cup f, P(w))$. But $w \in Ch(\hat{\mu}(f) \cup w, P(f))$ implies

$$w \in Ch(Ch(\mu(f) \cup \mu'(f), P(f)) \cup w, P(f)) = Ch(\mu(f) \cup \mu'(f) \cup w, P(f)).$$

By substitutability of $P(f)$, $w \in Ch(\mu(f) \cup w, P(f))$. Then $f \notin \mu(w)$ and $f \in Ch(\mu(w) \cup f, P(w))$ imply that (w, f) is also a pairwise block of μ.

So μ, $\mu' \in S(P)$ implies that there are no pairwise blocks of $\hat{\mu}$.

When $P(W)$ is substitutable, $S(P) = \mathscr{E}(P)$ and it is routine to verify that

$$\hat{\mu}(w) \subseteq V(w, \hat{\mu}) \subseteq V(w, \mu) \cap V(w, \mu').$$

Then μ, $\mu' \in \mathscr{E}(P)$ implies that $\mu(w)\, R(w)\, \hat{\mu}(w)$ and $\mu'(w)\, R(w)\, \hat{\mu}(w)$. □

Property $\hat{\mu}(w) \in \{\mu(w), \mu'(w)\}$ obviously holds in the many-to-one model. Thus Proposition 9.10 embeds Roth's result for the many-to-one model. Seemingly, the many-to-one-ness of the many-to-one model is behind Roth's result—we cannot capture the stronger coincidence-of-interest property in a many-to-many model with additional structure on preferences.

9.4.2 *Distributive property of lattice operations* The set of stable matchings in the one-to-one model is a distributive lattice (Knuth 1976). The distributive property of the one-to-one model does not extend to our many-to-many model: In Blair's (1988) Example 5.2, the set of stable many-to-many matchings is not a distributive lattice, and all agents' preferences in Blair's example satisfy strong substitutability (see Section 9.5).

We identify why the distributive property fails in the many-to-many model. The problem is that the lattice operations (see Remark 9.5) in $\langle \mathscr{V}, \leq \rangle$ may not preserve the property that matchings in $\mathscr{E}(P)$ are matchings—not only prematchings. That is, if $\mu \vee \mu' \in \mathscr{M}$ and $\mu \wedge \mu' \in \mathscr{M}$ for all μ and μ' in $\mathscr{E}(P)$, then $\langle \mathscr{E}(P), \leq_F \rangle$ is a distributive lattice. This result does extend the one-to-one result.

Let us order $\mathscr{V}$ by set-inclusion; let $\nu' \sqsubseteq \nu$ if $\nu'(f) \subseteq \nu(f)$ and $\nu(w) \subseteq \nu'(w)$ for all f and w. Then $\langle \mathscr{V}, \sqsubseteq \rangle$ is a lattice (see Remark 9.1). The lattice operations are $\sqcup$ and $\sqcap$, defined by $(\nu \sqcup \nu')(f) = \nu(f) \cup \nu'(f)$, and $(\nu \sqcap \nu')(f) = \nu(f) \cap \nu'(f)$ for all f, and $(\nu \sqcup \nu')(w) = \nu(w) \cap \nu'(w)$, and $(\nu \sqcap \nu')(w) = \nu(w) \cup \nu'(w)$ for all w.

Let $\psi : \mathscr{V} \to \mathscr{V}$ be defined by

$$(\psi \nu)(a) = \begin{cases} U(f, \nu) & \text{if } a = f \in F \\ V(w, \nu) & \text{if } a = w \in W. \end{cases}$$

THEOREM 9.11. *Let P be strongly substitutable. The map ψ is a lattice homomorphism of $\langle \mathscr{V}, \leq \rangle$ into $\langle \mathscr{V}, \sqsubseteq \rangle$. Further, if $\mu \vee \mu' \in \mathscr{M}$ and $\mu \wedge \mu' \in \mathscr{M}$ for all $\mu, \mu' \in \mathscr{E}(P)$, then*

(i) $\langle \mathscr{E}(P), \leq \rangle$ is a distributive sublattice of $\langle \mathscr{V}, \leq \rangle$

(ii) $\psi|_{\mathscr{E}(P)}$ is a lattice isomorphism of $\langle \mathscr{E}(P), \leq \rangle$ onto $\langle \psi \mathscr{E}(P), \sqsubseteq \rangle$.

See Section 14 for a proof.

The partial order $\leq$ on $\mathscr{V}$ depends on the profile P of preferences. In Theorem 9.11, we translate $\leq$ into an order that does not depend on P: set-inclusion. We interpret the result as showing how the lattice structure on $\mathscr{V}$ and, under additional assumptions, $\mathscr{E}(P)$, is inherited from the lattice structure of set inclusion.

The interest of Theorem 9.11 is, first, that it shows why distributivity fails in the many-to-many model. Second, it shows how the distributive property in the one-to-one model is inherited from the distributive property of set inclusion on $\mathscr{V}$; it is easy to verify that the one-to-one model satisfies that $\mu \vee \mu' \in \mathscr{M}$ and $\mu \wedge \mu' \in \mathscr{M}$ for all $\mu, \mu' \in \mathscr{E}(P)$. In fact the verification is carried out in Knuth (1976, page 56), as a first step in the proof that $\langle S(P), \leq_F \rangle$ is a distributive lattice.

Note that $\langle \mathscr{E}(P), \leq_F \rangle$ being a sub-lattice of $\langle \mathscr{V}, \leq_F \rangle$ means that the lattice operations $\vee_F$ and $\wedge_F$ on $\langle \mathscr{V}, \leq_F \rangle$ (see Remark 9.5) are also the lattice operations of $\langle \mathscr{E}(P), \leq_F \rangle$. Martínez et al. (2001), assuming substitutable (and q-separable) preferences, show that the stable matchings are not a lattice under $\vee_F$ and $\wedge_F$.

9.5 *Examples 5.1 and 5.2 in Blair (1988)*

We do not reproduce the examples here. We proceed to discuss the examples in light of our results.

Blair presents Example 5.1 as an example where $\langle S(P), \leq \rangle$ is not a lattice. In Example 5.1 there are 13 firms and 12 workers; $F = \{1, 2, \ldots, 13\}$, $W = \{a, b, \ldots, q\}$. Firm 10's preference relation is not strongly substitutable:

$$P(10): mp, bnp, m, \ldots,$$

where $\ldots$ means that there are other acceptable sets of workers not listed. Note that $\{b, n, p\}\, P(10)\, \{m\}$ and $b \in Ch(\{b, n, p\} \cup \{b\}, P(10))$, but that $b \in Ch(\{m\} \cup \{b\}, P(10)) = \{m\}$.

Thus Blair's Example 5.1 illustrates that, with non-strongly substitutable preferences, $\langle \mathscr{E}(P), \leq \rangle$ may not be a lattice.

Blair presents Example 5.2 as an example where $\langle S(P), \leq^B \rangle$ is not a distributive lattice. Preferences in Example 5.2 are strongly substitutable—this is easy, if tedious, to verify. Blair's example thus illustrates that $\langle \mathscr{E}(P), \leq \rangle$ and $\langle \mathscr{E}(P), \leq^B \rangle$ may not be distributive lattices (the lattice operations in $\langle \mathscr{E}(P), \leq \rangle$ and $\langle \mathscr{E}(P), \leq^B \rangle$ might not coincide, but in this example they do).

We show that Example 5.2 does not satisfy the property that $\mu \vee \mu' \in \mathscr{M}$ and $\mu \wedge \mu' \in \mathscr{M}$ for all $\mu, \mu' \in \mathscr{E}(P)$. So the example does not satisfy the hypotheses of Theorem 9.11.

In Example 5.2 there are 7 firms and 10 workers; $F=\{1,2,\ldots,7\}$, $W=\{a,b,\ldots,j\}$. Consider the matchings

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mu_1 & bcd & ae & af & j & h & i & g, \\ \mu_2 & bcd & ae & i & ag & h & f & j. \end{array}$$

Then $\mu_1 \vee \mu_2$ is

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \mu_1 \vee \mu_2 & bcd & ae & i & j & h & f & g \end{array}$$

$$\begin{array}{ccccccccccc} & a & b & c & d & e & f & g & h & i & j \\ \mu_1 \vee \mu_2 & 24 & 1 & 1 & 1 & 2 & 6 & 7 & 5 & 3 & 4. \end{array}$$

But $\mu_1 \vee \mu_2$ is not a matching, as $4 \in \mu_1 \vee \mu_2(a)$ while $a \notin \mu_1 \vee \mu_2(4)$.

Finally, we show that Blair's Example 5.2 also violates Roth's stronger conflict-of-interest property. From μ_1 and μ_2, constructing matching $\hat{\mu}$ by $\hat{\mu}(f)=Ch(\mu_1(f)\cup\mu_2(f), P(f))$ for all f gives

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hat{\mu} & bcd & ae & i & j & h & f & g. \end{array}$$

Now, $\hat{\mu}$ is blocked by the pair $(1,a)$, so $\hat{\mu}$ is not pairwise stable. Note that $\hat{\mu}(a)=\{2\}\notin \{\mu_1(a),\mu_2(a)\}$. Thus Example 5.2 does not satisfy the hypotheses of Proposition 9.10.

10. Summary

Table 1 summarizes our results. Each row shows results under some hypothesis on firms' preferences. Each column shows results under some restriction on workers' preferences. As one moves down and right on the table, the restrictions are stronger. Hence all results that hold in one entry, hold for all entries down and right.

Note that in the table we not denote the dependence of the solutions on the preference profile.

11. Proof of Theorem 6.1

The following proposition is immediate, but useful in some of our proofs.

Proposition 11.1. *A pair $(B,f)\in 2^W \times F$ blocks* μ if and only if, for all $w\in B$, there is $D_w \subseteq \mu(w)$ such that*

$$[D_w \cup f]\ P(w)\, \mu(w)$$

and there is $A\subseteq \mu(f)$ such that

$$[A\cup B]\ P(f)\, \mu(f).$$

In words, (B,f) blocks* μ if firm f is willing to hire the workers in B—possibly after firing some of its current workers in $\mu(f)$—and all workers w in B prefer f, possibly after rejecting some of the firms in $\mu(w)$.

$P(F)$	$P(W)$ Arbitrary	Substitutable	Strongly substitutable
Arbitrary	$\mathcal{E} \subseteq S^* \subseteq S$ $SW \subseteq \mathcal{E}$ $B \subseteq \mathcal{E}$	$\mathcal{E} = S^*$ $\mathcal{E} \subseteq C(P^B)$ $\mathcal{E} = SPNE$	No additional results
Substitutable		$\mathcal{E} = S$, $\mathcal{E} \neq \emptyset$ P^B strongly substituable $\langle \mathcal{E}, \leq^B \rangle$ a lattice T algorithm	$\mathcal{E} = SW = B$ $\mathcal{E} \subseteq IRC$ Roth (1985) opposition of interest (1)
Strongly substitutable			$\langle \mathcal{E}, \leq \rangle$ a lattice Roth (1985) opposition of interest (2) ψ lattice homomorphism Distributivity result

Table 1. Summary of Results. The empty entries in the table are completed by symmetry. An entry lists the results—in addition to the results above-and-left of the entry—we obtain under the corresponding restriction on preferences.

We present the proof of Theorem 6.1 in a series of lemmas. The first statement in Theorem 6.1 follows from Lemmas 11.2 and 11.5. Item (i) of the theorem follows from Lemmas 11.6 and 11.7. Item (ii) follows from Lemma 11.8, and Theorem 9.7.

Lemma 11.2. $S^*(P) \subseteq S(P)$.

Proof. Let $\mu \notin S(P)$. We now prove that $\mu \notin S^*(P)$. If μ is not individually rational there is nothing to prove; assume then that μ is individually rational. Because $\mu \notin S(P)$, there is $(w, f) \in W \times F$ such that $w \notin \mu(f)$ (or $f \notin \mu(w)$),

$$f \in Ch(\mu(w) \cup \{f\}, P(w)), \tag{1}$$

and

$$w \in Ch(\mu(f) \cup \{w\}, P(f)). \tag{2}$$

Statements (1), (2), and $w \notin \mu(f)$ imply that

$$Ch(\mu(w) \cup \{f\}, P(w))\; P(w)\; \mu(w) \tag{3}$$

and

$$Ch(\mu(f) \cup \{w\}, P(f))\; P(f)\; \mu(f). \tag{4}$$

Let $B = \{w\}$, $D_w = Ch(\mu(w) \cup \{f\}, P(w)) \cap \mu(w)$, and

$$A = Ch(\mu(f) \cup \{w\}, P(f)) \cap \mu(f).$$

We now prove that (B,f) blocks* μ. Since $B=\{w\}$, (3) implies that

$$\begin{aligned} D_w &= Ch(\mu(w)\cup\{f\},P(w))\cap\mu(w) \\ &= Ch(\mu(w)\cup\{f\},P(w))\setminus\{f\}. \end{aligned}$$

So (3) implies that

$$[D_w\cup\{f\}]=Ch(\mu(w)\cup\{f\},P(w))P(w)\mu(w),$$

which gives us the first part of the definition of block*. Also,

$$\begin{aligned} A &= Ch(\mu(f)\cup\{w\},P(f))\cap\mu(f) \\ &= Ch(\mu(f)\cup\{w\},P(f))\setminus\{w\} \\ &= Ch(\mu(f)\cup\{w\},P(f))\setminus B. \end{aligned}$$

So (4) implies that

$$[A\cup B]=Ch(\mu(f)\cup\{w\},P(f))\,P(f)\,\mu(f),$$

and we have the second part of the definition of block*. Thus, $\mu\notin S^*(P)$. □

Remark 11.3. In general, $S^*(P)\neq S(P)$.

We use Lemma 11.4 in many of our results, starting with Lemma 11.5.

Lemma 11.4. *If $\nu\in\mathscr{E}(P)$, then ν is a matching and ν is individually rational.*

Proof. Let $\nu=(\nu_F,\nu_W)\in\mathscr{E}(P)$.

Fix $w\in\nu_F(f)$. We prove that $f\in\nu_W(w)$. The condition $\nu\in\mathscr{E}(P)$ implies that

$$w\in\nu_F(f)=(T\nu)(f)=Ch(U(f,\nu),P(f)). \tag{5}$$

Thus $w\in U(f,\nu)$.

The definition of $U(f,\nu)$ implies

$$f\in Ch(\nu_W(w)\cup\{f\},P(w))\,R(w)\,\nu_W(w). \tag{6}$$

Now,

$$\begin{aligned} Ch(\nu_F(f),P(f)) &= Ch(Ch(U(f,\nu),P(f)),P(f)) && (7)\\ &= Ch(U(f,\nu),P(f)) && (8)\\ &= \nu_F(f). && (9) \end{aligned}$$

Equalities (7) and (9) follow from (5). Equality (8) is a simple property of choice sets: $Ch(Ch(S,P(f)),P(f))=Ch(S,P(f))$. Hence we have

$$\nu_F(f)=Ch(\nu_F(f),P(f)). \tag{10}$$

Now $w \in \nu_F(f)$ implies that $Ch(\nu_F(f), P(f)) = Ch(\nu_F(f) \cup \{w\}, P(f))$. So (10) implies that

$$f \in V(w, \nu). \tag{11}$$

But

$$\nu_W(w) = (T\nu)(w) = Ch(V(w, \nu), P(w)),$$

so

$$\nu_W(w) \subseteq V(w, \nu). \tag{12}$$

But (11) and (12) give

$$V(w, \nu) \supseteq \nu_W(w) \cup \{f\} \supseteq Ch(\nu_W(w) \cup \{f\}, P(w)).$$

The definition of choice set implies

$$\nu_W(w)\ R(w)\ Ch(\nu_W(w) \cup \{f\}, P(w)). \tag{13}$$

Statements (6), (13), and anti-symmetry of preference relations imply that $f \in \nu_W(w)$.

Let $f \in \nu_W(w)$. The proof that $w \in \nu_F(f)$ and

$$\nu_F(f) = Ch(\nu_F(f), P(f)) \tag{14}$$

is entirely symmetric with the proof for workers above.

Thus, $w \in \nu_F(f)$ if and only if $f = \upsilon_W(w)$. So, ν is a matching.

Statements (10) and (14) imply that ν is individually rational. □

LEMMA 11.5. $\mathscr{E}(P) \subseteq S^*(P)$.

PROOF. Let $\mu \in \mathscr{E}(P)$. By Lemma 11.4 we know that μ is an individually rational matching. Fix $f \in F$, $B \subseteq W$ such that $B \neq \emptyset$. We assume that for all $w \in B$ there exists $D_w \subseteq \mu(w)$ such that $\{f\} \cup D_w\ P(w)\ \mu(w)$. μ is individually rational, so $\mu(w) = Ch(\mu(w), P(w))$. Then $\{f\} \cup D_w P(w) \mu(w)$ implies that

$$f \in Ch(\mu(w) \cup \{f\}, P(w)) \tag{15}$$

for all $w \in B$. By the definition of $U(f, \mu)$, we have

$$B \subseteq U(f, \mu). \tag{16}$$

Let $A \subseteq \mu(f)$. The condition $\mu \in \mathscr{E}(P)$ implies that $\mu(f) = (T\mu)(f) \subseteq U(f, \mu)$, so (16) gives

$$A \cup B \subseteq U(f, \mu). \tag{17}$$

Now, $\mu \in \mathscr{E}(P)$ and (17) imply

$$\mu(f) R(f)\ Ch(A \cup B, P(f)) R(f) [A \cup B], \tag{18}$$

as $\mu(f) = Ch(U(f, \mu), P(f))$.

Statements (15) and (18) show that there is no (B, f) that blocks* μ. The proof that there is no $(w, A) \in W \times 2^F$ that blocks* μ is symmetric. Thus $\mu \in S^*(P)$. □

LEMMA 11.6. *If $P(W)$ is substitutable, then $S^*(P) \subseteq \mathscr{E}(P)$.*

PROOF. Let $\mu \in S^*(P)$ and assume that $\mu \notin \mathscr{E}(P)$, so $\mu \neq T\mu$. We first prove that $(T\mu)(f) \neq \mu(f)$ for some f yields a contradiction, and then that $(T\mu)(w) \neq \mu(w)$ for some w yields a contradiction. Note that, by the asymmetric situation of firms and workers in the definition of $S^*(P)$, the proofs of the two statements are not analogous.

First assume that there exist $f \in F$ such that

$$\mu(f) \neq (T\mu)(f) = Ch(U(f,\mu), P(f)) = C \subseteq U(f,\mu).$$

Let $A = C \cap \mu(f)$ and $B = C \setminus \mu(f)$. Because μ is an individually rational matching we have $\mu(w) = Ch(\mu(w), P(w)) = Ch(\mu(w) \cup f, P(w))$ for all $w \in \mu(f)$. Hence $\mu(f) \subseteq U(f,\mu)$, so $(T\mu)(f)\, P(f)\, \mu(f)$ implies that $B \neq \emptyset$.

Now,

$$A \cup B = C\ P(f)\ \mu(f). \tag{19}$$

Also, for all $w \in B$, $w \in U(f,\mu)$; so $f \in Ch(\mu(w) \cup f, P(w))$ by the definition of $U(f,\mu)$. For any $w \in B$, let $D_w = Ch(\mu(w) \cup f, P(w)) \cap \mu(w)$. Since $f \notin \mu(w)$ we have

$$\{f\} \cup D_w\ P(w)\ \mu(w). \tag{20}$$

Statements (19) and (20) imply that (B, f) blocks* μ, which contradicts $\mu \in S^*(P)$.

Hence, for all $f \in F$,

$$\mu(f) = (T\mu)(f). \tag{21}$$

Now assume that there exists $w \in W$ such that

$$\mu(w) \neq (T\mu)(w) = Ch(V(w,\mu), P(w)) = G \subseteq V(w,\mu).$$

If $f \in G$, then

$$w \in Ch(\mu(f) \cup \{w\}, P(f)) \tag{22}$$

by the definition on $V(w,\mu)$. Because μ is an individually rational matching we have—by the same argument as above—$\mu(w) \subseteq V(w,\mu)$. We can assume that $G \nsubseteq \mu(w)$; for, if $G \subseteq \mu(w)$, then $\mu(w) \subseteq V(w,\mu)$ and the Choice Property[7] imply that

$$G = Ch(V(w,\mu), P(w)) = Ch(\mu(w), P(w)) = \mu(w),$$

where the last equality follows because μ is an individually rational matching—but this would contradict $G \neq \mu(w)$, hence we can assume $G \nsubseteq \mu(w)$.

Let $\overline{f} \in G \setminus \mu(w)$. Because μ is a matching, $w \notin \mu(\overline{f})$. Now, (22) implies that

$$w \in Ch(\mu(\overline{f}) \cup \{w\}, P(\overline{f})) = C.$$

Let $A = C \cap \mu(\overline{f}) = C \setminus \{w\}$ and $B = \{w\}$. Then

$$C = [A \cup B]\ P(\overline{f})\ \mu(\overline{f}). \tag{23}$$

[7] If $Ch(A, P(s)) \subseteq B \subseteq A$, then $Ch(A, P(s)) = Ch(B, P(s))$.

Now, $\overline{f} \in G \setminus \mu(w)$, so that the substitutability of $P(w)$ implies that there exists $D_w = Ch(V(w,\mu), P(w)) \cap \mu(w)$ such that

$$[\overline{f} \cup D_w]\ P(w)\ \mu(w). \tag{24}$$

Statements (23) and (24) imply that $(\overline{f}, \{w\})$ blocks* μ, which contradicts $\mu \in S^*(P)$. Hence, for all $w \in W$,

$$\mu(w) = (T\mu)(w). \tag{25}$$

Statements (21) and (25) imply that $\mu = T\mu$. Hence $\mu \in \mathscr{E}(P)$. □

LEMMA 11.7. *If $P(W)$ is substitutable then $S^*(P) \subseteq C(P^B)$.*

PROOF. Let $\mu \in S^*(P)$ and suppose that $\mu \notin C(P^B)$. Let $F' \subseteq F$, $W' \subseteq W$ with $F' \cup W' \neq \emptyset$, and $\hat{\mu} \in \mathscr{M}$ such that, for all $w \in W'$, and for all $f \in F'$

$$\hat{\mu}(w) \subseteq F' \quad \text{and} \quad \hat{\mu}(f) \subseteq W' \tag{26}$$

$$\hat{\mu}(w)\ R^B(w)\ \mu(w) \tag{27}$$

$$\hat{\mu}(f)\ R^B(f)\ \mu(f)$$

$$\hat{\mu}(s)\ P^B(s)\ \mu(s) \text{ for at least one } s \in W' \cup F'.$$

We need the following.

CLAIM. *There exists $f \in F'$ such that $\hat{\mu}(f)\ P^B(f)\ \mu(f)$ if and only if there exists $w \in W'$ such that $\hat{\mu}(w)\ P^B(w)\ \mu(w)$.*

PROOF OF CLAIM. Let $\hat{\mu}(f)\ P^B(f)\ \mu(f)$. Because μ is individually rational, we have $\hat{\mu}(f) \nsubseteq \mu(f)$, so let $\overline{w} \in \hat{\mu}(f) \setminus \mu(f)$. By (26), we have $\overline{w} \in \hat{\mu}(f) \subseteq W'$; but then $\overline{w} \notin \mu(f)$ and (27) imply that

$$\hat{\mu}(\overline{w})\ P^B(\overline{w})\ \mu(\overline{w}).$$

Similarly we show that if $\hat{\mu}(w)\ P^B(w)\ \mu(w)$ then there exists $\overline{f}$ such that $\hat{\mu}(\overline{f}) P^B(\overline{f})\ \mu(\overline{f})$. □

By the claim, we can assume that there exists $f \in F'$ such that $\hat{\mu}(f) \neq \mu(f)$. Let $B = \hat{\mu}(f) \setminus \mu(f)$ and $A = \hat{\mu}(f) \cap \mu(f)$. Then

$$A \cup B = \hat{\mu}(f)\ P^B(f)\ \mu(f) \tag{28}$$

and $B \cap \mu(f) = \emptyset$. Let $w \in B$. Then $f \in \hat{\mu}(w)$ and $f \notin \mu(w)$, which imply that $\hat{\mu}(w) \neq \mu(w)$. Condition (27) implies that

$$f \in \hat{\mu}(w) = Ch(\mu(w) \cup \hat{\mu}(w), P(w)).$$

By substitutability of $P(w)$, $f \in Ch(\mu(w) \cup f, P(w))$.

Now, $w \in B$ was arbitrary, so together with (28), this implies that (B, f) blocks* μ. Thus $\mu \notin S^*(P)$. □

LEMMA 11.8. *If P is substitutable then $S(P) \subseteq \mathscr{E}(P)$.*

PROOF. Let $\mu \notin \mathscr{E}(P)$. We prove that $\mu \notin S(P)$. If μ is not individually rational there is nothing to prove. Suppose then that μ is individually rational. Lemma 11.6 and $\mu \notin \mathscr{E}(P)$ imply $\mu \notin S^*(P)$. So, there is (B, f) with $B \neq \emptyset$ that blocks* μ. This means that, for all $w \in B$,

$$f \in Ch(\mu(w) \cup f, P(w))$$

and

$$B \subseteq Ch(\mu(f) \cup B, P(f)).$$

But $P(f)$ is substitutable, so there is $w' \in B$ with

$$w' \in Ch(\mu(f) \cup w', P(f)).$$

Thus $\mu \notin S(P)$. □

12. PROOFS OF THEOREMS 6.4 AND 6.5

The proof of Theorem 6.4 follows from Lemmas 12.1, 12.2, 12.3, 12.4, and 12.5. Theorem 6.5 then follows from Theorem 9.7.

LEMMA 12.1. *$SW(P) \subseteq \mathscr{E}(P)$.*

PROOF. Let μ be a matching such that $\mu \notin \mathscr{E}(P)$. We prove that $\mu \notin SW(P)$. If μ is not individually rational there is nothing to prove. Suppose then that μ is an individually rational matching.

Suppose, without loss of generality, that there is $\overline{f} \in F$ such that $\mu(\overline{f}) \neq Ch(U(\overline{f}, \mu), P(\overline{f}))$. That μ is individually rational implies that $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ since, for all $w \in \mu(\overline{f})$, $\overline{f} \in \mu(w)$, so

$$Ch(\mu(w) \cup \overline{f}, P(w)) = Ch(\mu(w), P(w)) = \mu(w) \ni \overline{f}.$$

Let $F' = \{\overline{f}\}$ and $W' = Ch(U(\overline{f}, \mu), P(\overline{f})) \setminus \mu(\overline{f})$. We construct $\mu' \in \mathscr{M}$ such that $\langle W', F', \mu' \rangle$ is a setwise block of μ. Let $\mu'(\overline{f}) = Ch(U(\overline{f}, \mu), P(\overline{f}))$. For all $w \in W$, let

$$\mu'(w) = \begin{cases} Ch(\mu(w) \cup \overline{f}, P(w)) & \text{if } w \in W' \\ \mu(w) & \text{if } w \in \left[\mu'(\overline{f}) \cap \mu(\overline{f})\right] \cup \left[\mu'(\overline{f}) \cup \mu(\overline{f})\right]^c \\ \mu(w) \setminus \overline{f} & \text{if } w \in \mu(\overline{f}) \setminus \mu'(\overline{f}). \end{cases}$$

The matching $\mu'(f)$, for $f \notin F'$, is determined from the $\mu'(w)$'s. Then μ' is a matching and $W' = \mu'(F) \setminus \mu(F)$. Note that $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ implies that $\overline{f} \in Ch(\mu(w) \cup \overline{f}, P(w))$, so $\overline{f} \in \mu'(w)$ for all $w \in W'$. So $F' = \mu'(W) \setminus \mu(W)$.

First we verify that μ' is individually rational: $\mu'(\overline{f}) = Ch(\mu'(\overline{f}), P(\overline{f}))$, as $\mu'(\overline{f}) = Ch(U(\overline{f}, \mu), P(\overline{f}))$; and $\mu'(w) = Ch(\mu'(w), P(\overline{f}))$, as $\mu'(w) = Ch(\mu(w) \cup \overline{f}, P(\overline{f}))$ for all $w \in W'$.

Finally, $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ implies that $\mu'(\overline{f})\ P(\overline{f})\ \mu(\overline{f})$ and $\mu'(w) = Ch(\mu(w) \cup \overline{f}, P(\overline{f}))$ implies that $\mu'(w)\ P(w)\ \mu(w)$, for all $w \in W'$. Thus the constructed $\langle W', F', \mu' \rangle$ is a setwise block of μ, and thus $\mu \notin SW(P)$. □

LEMMA 12.2. $B(P) \subseteq \mathscr{E}(P)$.

PROOF. Let μ be a matching such that $\mu \notin \mathscr{E}(P)$. We prove that $\mu \notin B(P)$. If μ is not individually rational there is nothing to prove. Suppose then that μ is individually rational.

Suppose, without loss of generality, that there is $\overline{f} \in F$ such that $\mu(\overline{f}) \neq Ch(U(\overline{f}, \mu), P(\overline{f}))$. Let $F' = \{\overline{f}\}$ and

$$W' = Ch(U(\overline{f}, \mu), P(\overline{f})) \setminus \mu(\overline{f}).$$

Construct μ' as in the proof of Lemma 12.1. We prove that the objection $\langle W', F', \mu' \rangle$ is counterobjection-free. Recall that

$$\mu'(\overline{f}) = Ch(U(\overline{f}, \mu), P(\overline{f})). \tag{29}$$

Note that $\mu(\overline{f}) \subseteq U(\overline{f}, \mu)$ (see the proof of Lemma 12.1). So (29) implies $\mu'(\overline{f})\ P(\overline{f})\ \mu(\overline{f})$. Similarly, $W' \subseteq U(\overline{f}, \mu)$ implies that $\mu'(w) = Ch(\mu(w) \cup \overline{f})\ P(w)\ \mu(w)$ for all $w \in W'$. Hence $\langle W', F', \mu' \rangle$ is an objection.

We now prove that there are no counterobjections to $\langle W', F', \mu' \rangle$. Let $\langle W'', F'', \mu'' \rangle$ be such that μ'' is a matching, $F'' \subseteq F'$, $W'' \subseteq W'$, and $\mu''(W'' \cup F'') \setminus \mu'(W'' \cup F'') \subseteq W'' \cup F''$.

First, let $F'' \neq \emptyset$. Then $F'' = \{\overline{f}\}$. Statement (29) implies that $\mu'(\overline{f})\ R(\overline{f})\ A$ for all $A \subseteq \mu'(\overline{f})$. But

$$\mu''(\overline{f}) \setminus \mu'(\overline{f}) \subseteq W' \subseteq \mu'(\overline{f}).$$

So $\mu'(\overline{f})\ R(\overline{f})\ \mu''(\overline{f})$. Thus $\langle W'', F'', \mu'' \rangle$ is not a counterobjection.

Second, let $F'' = \emptyset$. For all $w \in W''$, $w \in U(\overline{f}, \mu)$. So

$$\overline{f} \in Ch(\mu(w) \cup \overline{f}, P(w)) = \mu'(w) \tag{30}$$

(see the proof of Lemma 12.1). Also, $\mu''(w) \subseteq \mu'(w)$, as $\mu''(W'') \setminus \mu'(W'') \subseteq F'' \neq \emptyset$. Then $\overline{f} \notin \mu''(w)$ and (30) imply that $\mu'(w)\ P(w)\ \mu''(w)$. Thus $\langle W'', F'', \mu'' \rangle$ is not a counterobjection. □

LEMMA 12.3. *If $P(F)$ is substitutable and $P(W)$ is strongly substitutable, then $\mathscr{E}(P) \subseteq SW(P)$.*

PROOF. The proof is similar to the proof of Lemma 12.5. Let $\mu \in \mathscr{E}(P)$. By Lemma 11.4, μ is an individually rational matching. Suppose, by way of contradiction, that $\mu \notin SW(P)$. Let $\langle W', F', \mu' \rangle$ be a setwise block of μ.

Fix $\overline{f} \in F'$, so $\mu'(\overline{f})\ P(\overline{f})\ \mu(\overline{f})$. The matching μ is individually rational, so $\mu'(\overline{f})\ P(\overline{f})\ \mu(\overline{f})$ implies that

$$Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f})) \nsubseteq \mu(\overline{f}).$$

Fix $\overline{w} \in Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f}))$ such that $\overline{w} \in \mu'(\overline{f}) \setminus \mu(\overline{f})$. By substitutability of $P(\overline{f})$, $\overline{w} \in Ch(\mu(\overline{f}) \cup \overline{w}, P(\overline{f}))$. So

$$\overline{f} \in V(\overline{w}, \mu). \tag{31}$$

On the other hand, $\overline{w} \in \mu'(\overline{f}) \setminus \mu(\overline{f})$ implies that $\overline{w} \in W'$, so

$$\mu'(\overline{w})\ P(\overline{w})\ \mu(\overline{w}). \tag{32}$$

Now, $\langle W', F', \mu' \rangle$ is a setwise block, so $\mu'(\overline{w}) = Ch(\mu'(\overline{w}), P(\overline{w}))$. Further, μ' is a matching, so $\overline{f} \in \mu'(\overline{w})$. Then $\overline{f} \in Ch(\mu'(\overline{w}) \cup \overline{f}, P(\overline{w}))$. Strong substitutability of $P(\overline{w})$ and (32) imply that

$$\overline{f} \in Ch(\mu(\overline{w}) \cup \overline{f}, P(\overline{w})). \tag{33}$$

But (31) and $\mu \in \mathscr{E}(P)$ imply that $\mu(\overline{w}) \cup \overline{f} \subseteq V(\overline{w}, \mu)$. But then (33) contradicts $\mu(\overline{w}) = Ch(V(\overline{w}, \mu), P(\overline{w}))$. □

LEMMA 12.4. *If $P(F)$ is substitutable and $P(W)$ is strongly substitutable, then $\mathscr{E}(P) \subseteq B(P)$.*

PROOF. The proof is similar to the proof of Lemma 12.5. Let $\mu \in \mathscr{E}(P)$. By Lemma 11.4, μ is an individually rational matching. Let $\langle W', F', \mu' \rangle$ be an objection to μ.

First, if $\mu'(s) \neq Ch(\mu'(s), P(s))$ for some $s \in F' \cup W'$, then $\langle W', F', \mu' \rangle$ has a counter-objection: let $\overline{f} \in F'$ be such that $\mu'(\overline{f}) \neq Ch(\mu'(\overline{f}), P(\overline{f}))$. Let $W'' = \emptyset$, $F'' = \{\overline{f}\}$, and let μ'' be defined by $\mu''(f) = \mu'(f)$ for all $f \neq \overline{f}$ and $\mu''(\overline{f}) = Ch(\mu'(\overline{f}), P(\overline{f}))$. The definition of $\mu''(w)$, for all $w \in W$, is implicit. Then $\mu'(\overline{f}) \neq Ch(\mu'(\overline{f}), P(\overline{f}))$ implies that $\mu''(\overline{f})\, P(\overline{f})\, \mu'(\overline{f})$. Hence $\langle W'', F'', \mu'' \rangle$ is a counterobjection to $\langle W', F', \mu' \rangle$. So $\mu \in B(P)$.

Second, let $\mu'(s) = Ch(\mu'(s), P(s))$ for all $s \in F' \cup W'$. We prove that $\langle W', F', \mu' \rangle$ cannot be an objection. Suppose, by the way of contradiction, that $\langle W', F', \mu' \rangle$ is an objection to μ, so we can suppose—without loss of generality—that there is $\overline{f} \in F'$ such that $\mu'(\overline{f})\, P(\overline{f})\, \mu(\overline{f})$. The matching μ is individually rational, so $\mu'(\overline{f})\, P(\overline{f})\, \mu(\overline{f})$ implies that

$$\mu(\overline{f}) \nsubseteq Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f})).$$

Let $\overline{w} \in Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f}))$ be such that $\overline{w} \in \mu'(\overline{f}) \setminus \mu(\overline{f})$. Now, substitutability of $P(\overline{f})$ implies that

$$\overline{w} \in Ch(\mu(\overline{f}) \cup \overline{w}, P(\overline{f})).$$

Thus, $\overline{f} \in V(\overline{w}, \mu)$.

On the other hand, $\overline{w} \in \mu'(\overline{f}) \setminus \mu(\overline{f})$ implies that $\overline{w} \in W'$. So $\mu'(\overline{w})\ P(\overline{w})\ \mu(\overline{w})$, as $\langle W', F', \mu' \rangle$ is an objection. Then

$$\overline{f} \in \mu'(\overline{w}) = Ch(\mu'(\overline{w}), P(\overline{w})) = Ch(\mu'(\overline{w}) \cup \overline{f}, P(\overline{w}))$$

and strong substitutability of $P(\overline{w})$ give $\overline{f} \in Ch(\mu(\overline{w}) \cup \overline{f}, P(\overline{w}))$. But we proved that $\overline{f} \in V(\overline{w}, \mu)$. So

$$\mu(\overline{w}) \neq Ch(\mu(\overline{w}) \cup \overline{f}, P(\overline{w})) \subseteq V(\overline{w}, \mu).$$

This condition is a contradiction, since $\mu \in \mathscr{E}(P)$ implies that $\mu(\overline{w}) = Ch(V(\overline{w}, \mu), P(\overline{w}))$. Thus $\langle W', F', \mu' \rangle$ is not an objection, and $\mu \in B(P)$. □

LEMMA 12.5. *If $P(F)$ is substitutable and $P(W)$ is strongly substitutable, then $\mathscr{E}(P) \subseteq IRC(P)$.*

PROOF. Let $\mu \in \mathscr{E}(P)$. By Lemma 11.4, μ is an individually rational matching. Suppose, by way of contradiction, that $\langle W', F', \mu' \rangle$ is an individually rational block of μ.

Without loss of generality, let $\mu'(\overline{f})\, P(\overline{f})\, \mu(\overline{f})$, for some $\overline{f} \in F'$. Since μ is individually rational,

$$Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f})) \nsubseteq \mu(\overline{f}).$$

Let $\overline{w} \in \mu'(\overline{f}) \setminus \mu(\overline{f})$ be such that

$$\overline{w} \in Ch(\mu(\overline{f}) \cup \mu'(\overline{f}), P(\overline{f})) = Ch(\mu(\overline{f}) \cup \mu'(\overline{f}) \cup \overline{w}, P(\overline{f})).$$

By substitutability of $P(\overline{f})$, $\overline{w} \in Ch(\mu(\overline{f}) \cup \overline{w}, P(\overline{f}))$. Thus

$$\overline{f} \in V(\overline{w}, \mu). \tag{34}$$

Now, $\overline{f} \in \mu'(\overline{w}) \setminus \mu(\overline{w})$ implies

$$\mu'(\overline{w})\, P(\overline{w})\, \mu(\overline{w}), \tag{35}$$

as $\overline{w} \in W'$ and $\mu'(\overline{w}) \neq \mu(\overline{w})$. But μ' is individually rational, so

$$\overline{f} \in Ch(\mu'(\overline{w}), P(\overline{w})) = Ch(\mu'(\overline{w}) \cup \overline{f}, P(\overline{w})).$$

Then (35) and strong substitutability of $P(\overline{w})$ imply that $\overline{f} \in Ch(\mu(\overline{w}) \cup \overline{f}, P(\overline{w}))$. So

$$Ch(\mu(\overline{w}) \cup \overline{f}, P(\overline{w}))\, P(\overline{w})\, \mu(\overline{w}). \tag{36}$$

But $\mu \in \mathscr{E}(P)$ implies that $\mu(\overline{w}) = Ch(V(\overline{w}, \mu), P(\overline{f}))$. By (34), $\mu(\overline{w}) \cup \overline{f} \subseteq V(\overline{w}, \mu)$, which contradicts (36). Hence there are no individually rational blocks of μ, and $\mu \in IRC(P)$. □

13. Proofs of Theorems 7.1 and 8.2

13.1 *Proof of Theorem 7.1*

The proof of Theorem 7.1 follows from Lemmas 13.1 and 13.2.

LEMMA 13.1. *Let $P(W)$ be substitutable. If $\mu \in \mathscr{M}$ is the outcome of an SPNE, then $\mu \in \mathscr{E}(P)$.*

PROOF. Let (η^*, ξ^*) be an SPNE, and let $\mu \in \mathscr{M}$ be the outcome of (η^*, ξ^*). For all $w \in W$,

$$\xi^*_w(\eta) \cap \{f : w \in \eta_f\} = Ch(\{f : w \in \eta_f\}, P(w)).$$

For all $f \in F$, and all η_{-f}, let

$$Y(\eta_{-f}) = \{w : f \in Ch(\tilde{f} : w \in \eta_{\tilde{f}} \cup f, P(w))\}.$$

So, by definition of SPNE, $\eta^*_f \cap Y(\eta^*_{-f}) = Ch(Y(\eta^*_{-f}), P(f))$.

Let $(\overline{\eta},\overline{\xi})$ be the pair of strategies obtained from (η^*,ξ^*) by having each w not propose to firms that did not propose to w and each f not propose to workers who will reject f. Thus, $\overline{\xi}_w(\eta)=\xi^*_w(\eta)\cap\{f:w\in\eta_f\}$ and $\overline{\eta}_f\cap Y(\eta^*_{-f})=\eta^*_f\cap Y(\eta^*_{-f})$.

We show that $(\overline{\eta},\overline{\xi})$ is an SPNE as well, and that its outcome is also μ. First, it is immediate that its outcome is μ: $\overline{\eta}_f=\mu(f)$ for all f, and for all $w\in\mu(f)$, $f\in\overline{\xi}_w(\overline{\eta})$. Second, given a strategy profile η for firms, each w is indifferent between proposing $\xi^*_w(\eta)$ and $\overline{\xi}_w(\eta)$, as they both result in the same set of partners. For a firm f, $Y(\eta^*_{-f})=Y(\overline{\eta}_{-f})$, which implies that $\overline{\eta}_f=Ch(Y(\overline{\eta}_{-f}),P(f))$, and thus $(\overline{\eta},\overline{\xi})$ is an SPNE. To see that $Y(\eta^*_{-f})=Y(\overline{\eta}_{-f})$, note that $w\in Y(\eta^*_{-f})$ if and only if $f\in Ch(\{\tilde{f}:w\in\eta^*_{\tilde{f}}\}\cup f,P(w))$. But

$$\begin{aligned}Ch(\{\tilde{f}:w\in\eta^*_{\tilde{f}}\}\cup f,P(w))&=Ch(Ch(\{\tilde{f}:w\in\eta^*_{\tilde{f}}\},P(w))\cup f,P(w))\\&=Ch(\mu(w)\cup f,P(w))\\&=Ch(\{\tilde{f}:w\in\overline{\eta}_{\tilde{f}}\}\cup f,P(w)),\end{aligned}$$

where the first equality is a consequence of substitutability of $P(W)$ (Blair 1988, Proposition 2.3). Hence $w\in Y(\eta^*_{-f})$ if and only if

$$f\in Ch(\{\tilde{f}:w\in\overline{\eta}_{\tilde{f}}\}\cup f,P(w)),$$

so $Y(\eta^*_{-f})=Y(\overline{\eta}_{-f})$.

Now we prove that $\mu\in\mathscr{E}(P)$. Let $f\in F$. Note that

$$Y(\overline{\eta}_{-f})=\{w:f\in Ch(\mu(w)\cup f,P(w))\},$$

so $Y(\overline{\eta}_{-f})=U(f,\mu)$. Now, by the definition of $\overline{\eta}_f$, $\mu(f)=\overline{\eta}_f=Ch(U(f,\mu),P(w))$.

Let $w\in W$. We first prove that

$$\mu(w)\subseteq V(w,\mu). \tag{37}$$

Let $f\in\mu(w)$, so $w\in\mu(f)=\overline{\eta}_f$. But $\overline{\eta}_f=Ch(Y(\overline{\eta}_{-f}),P(f))$, so $\overline{\eta}_f=Ch(\overline{\eta}_f,P(f))$. Then $w\in Ch(\mu(f)\cup w,P(f))=Ch(\overline{\eta}_f,P(f))$, so $f\in V(w,\mu)$; this proves (37). Second, we prove that $Ch(V(w,\mu),P(w))\subseteq\mu(w)$, which together with (37) implies that $\mu(w)=Ch(V(w,\mu),P(w))$. Let $f\in Ch(V(w,\mu),P(w))$. By (37), $\mu(w)\cup f\subseteq V(w,\mu)$. Substitutability of $P(w)$ implies that

$$f\in Ch(\mu(w)\cup f,P(w)). \tag{38}$$

So $w\in U(f,\mu)$. Suppose, by way of contradiction, that $f\notin\mu(w)$. Now, $f\notin\mu(w)$ implies $w\notin\mu(f)$, so (38) implies $\mu(f)\cup f\ P(w)\ \mu(f)$. But $w\in U(f,\mu)$, so $\mu(f)\cup f\ P(w)\ \mu(f)$ contradicts $\mu(f)=\overline{\eta}_f=Ch(U(f,\mu),P(w))$. The assumption $f\notin\mu(w)$ is then absurd. This finishes the proof that $\mu(w)=Ch(V(w,\mu),P(w))$. We have also proved $\mu(f)=Ch(U(f,\mu),P(w))$, so $\mu\in\mathscr{E}(P)$. □

LEMMA 13.2. *If $\mu \in \mathscr{E}(P)$, then μ is the outcome of some SPNE.*

PROOF. Define $(\overline{\eta}, \overline{\xi})$ by $\overline{\eta}_f = \mu(f)$ and $\overline{\xi}_w(\eta) = Ch(\{f : w \in \eta_f\}, P(w))$. Let $\overline{\mu} \in \mathscr{M}$ be the outcome of $(\overline{\eta}, \overline{\xi})$. We show that $(\overline{\eta}, \overline{\xi})$ is an SPNE, and that $\overline{\mu} = \mu$.

Note that, for any f and w, $\{\tilde{f} : w \in \overline{\eta}_{\tilde{f}}\} \cup f = \mu(w) \cup f$. Then

$$\begin{aligned}\{w : f \in Ch(\{\tilde{f} : w \in \overline{\eta}_{\tilde{f}}\} \cup f, P(w))\} &= \{w : f \in Ch(\mu(w) \cup f, P(w))\} \\ &= U(f, \mu).\end{aligned}$$

But $\mu \in \mathscr{E}(P)$, so $\overline{\eta}_f = \mu(f) = Ch(U(f, \mu), P(f))$. Hence $\overline{\eta}_f$ is optimal given $\overline{\eta}_{-f}$. By definition of $\overline{\xi}_w$, $\overline{\xi}_w(\eta)$ is optimal for w given any profile η. Hence $(\overline{\eta}, \overline{\xi})$ is an SPNE.

Now, $f \in \mu(w)$ if and only if $w \in \mu(f) = \overline{\eta}_f$. So $\{f : w \in \overline{\eta}_f\} = \mu(w)$. Then $\overline{\xi}_w(\overline{\eta}) = Ch(\mu(w), P(w)) = \mu(w)$, as $\mu(w) \in \mathscr{E}(P)$ implies that μ is individually rational (Lemma 11.4).

Hence $w \in \overline{\mu}(f)$ if and only if $w \in \overline{\eta}_f = \mu(f)$, and $f \in \overline{\mu}(w)$ if and only if $f \in \overline{\xi}_w(\overline{\eta}) = \mu(w)$. So $\mu = \overline{\mu}$. □

13.2 *Proof of Theorem 8.2*

Let $\langle W', F', \mu' \rangle$ be a block of μ. Let $w \in W'$ be such that $\mu'(w)\, P(w)\, \mu(w)$. We prove that there are $f, f' \in F'$ and $w' \in W'$ such that:

- $w \neq w'$, $f \neq f'$
- $f \in \mu'(w) \setminus \mu(w)$, $w' \in \mu'(f) \setminus \mu(f)$, and $f' \in \mu'(w') \setminus \mu(w')$
- f wants to add w' and w' wants to add f', but w' does not want to add f and f' does not want to add w'.

Now, $\mu'(w)\, P(w)\, \mu(w)$ implies that

$$Ch(\mu(w) \cup \mu'(w), P(w))\, R(w)\, \mu'(w)\, P(w)\, \mu(w).$$

But $\mu \in \mathscr{E}(P)$ implies that μ is individually rational (Lemma 11.4); so $\mu(w)\, R(w)\, A$ for all $A \subseteq \mu(w)$. Hence

$$Ch(\mu(w) \cup \mu'(w), P(w)) \setminus \mu(w) \neq \emptyset.$$

Let $f \in Ch(\mu(w) \cup \mu'(w), P(w)) \setminus \mu(w)$. By substitutability of $P(w)$, $f \in Ch(\mu(w) \cup f, P(w))$; hence w wants to add f.

On the other hand, $f \in Ch(\mu(w) \cup f, P(w))$ implies that $w \in U(f, \mu)$. But $f \in \mu'(w) \setminus \mu(w)$ means that $w \in \mu'(f) \setminus \mu(f)$. In particular, $w \notin \mu(f)$; so, by Lemmas 11.5 and 11.2, $w \notin Ch(\mu(f) \cup w, P(f)) = \mu(f)$, as $\mu \in \mathscr{E}(P)$ implies $\mu(f) = Ch(U(f, \mu), P(f))$ and $\mu(f) \cup w \subseteq U(f, \mu)$. Hence f does not want to add w.

But $\mu'(f) \neq \mu(f)$ and $f \in F'$ imply $\mu'(f)\, P(f)\, \mu(f)$. By an argument that is symmetric to the one above, there is $w' \in \mu'(f) \setminus \mu(f)$ and $f' \in \mu'(w) \setminus \mu(w)$ such that f wants to add w' and w' wants to add f', but w' does not want to add f, and f' does not want to add w'.

Recursively, given $w_k \in W'$ with $\mu'(w_k)\,P(w_k)\,\mu(w_k)$ let f_{k+1}, w_{k+1}, and f_{k+1} be f, w', and f' obtained as above. Consider the sequence of alternating workers and firms constructed: W' is a finite set, so there must exist k and l such that $w_k = w_l$. Say $l < k$; set $\hat{w}_0 = w_l$, and $(\hat{w}_{k'}, \hat{f}_{k'}) = (w_{k'+l}, f_{k'+l})$ for $k' = 0, 1, \ldots, k-l$. The resulting sequence is an acceptance-rejection cycle for μ.

14. Proofs of Theorems 9.7, 9.8, and 9.11

14.1 *Proof of Theorem 9.7*

We first establish some lemmas.

Lemma 14.1. *Let P be substitutable. Let μ and μ' be pre-matchings. If $\mu \leq^B \mu'$ then, for all $w \in W$ and $f \in F$, $U(f,\mu) \subseteq U(f,\mu')$ and $V(w,\mu) \supseteq V(w,\mu')$.*

Proof. We prove that $V(w,\mu) \supseteq V(w,\mu')$. The proof that $U(f,\mu) \subseteq U(f,\mu')$ is analogous.

First, if $V(w,\mu') = \{\emptyset\}$, then there is nothing to prove, as $\{\emptyset\} = V(w,\mu') \subseteq V(w,\mu)$. Suppose that $V(w,\mu') \neq \{\emptyset\}$, and let $f \in V(w,\mu')$. Then, $w \in Ch(\mu'(f) \cup w, P(f))$.

But $\mu \leq^B \mu'$, so the definition of $\leq^B$ implies that, for all $f \in F$, either $\mu'(f) = \mu(f)$ so $w \in Ch(\mu(f) \cup w, P(f))$, or $\mu'(f) = Ch(\mu'(f) \cup \mu(f), P(f))$. Then $w \in Ch(\mu'(f) \cup w, P(f))$ implies that

$$\begin{aligned} w &\in Ch(\mu'(f) \cup w, P(f)) \\ &= Ch(Ch(\mu'(f) \cup \mu(f), P(f)) \cup w, P(f)) \\ &= Ch(\mu'(f) \cup \mu(f) \cup \{w\}, P(f)). \end{aligned}$$

The second equality above is from Proposition 2.3 in Blair (1988). (Blair proves that if P is substitutable, then $Ch(A \cup B, P(f)) = Ch(Ch(A, P(f)) \cup B, P(f))$ for all A and B.) Substitutability of P implies that $w \in Ch(\mu(f) \cup w, P(f))$. Then $f \in V(w,\mu)$ and thus $V(w,\mu) \supseteq V(w,\mu')$. □

Lemma 14.2. *Let P be strongly substitutable. Let μ and μ' be pre–matchings. If $\mu \leq \mu'$ then, for all $w \in W$ and $f \in F$, $U(f,\mu) \subseteq U(f,\mu')$ and $V(w,\mu) \supseteq V(w,\mu')$.*

Proof. We prove that $V(w,\mu) \supseteq V(w,\mu')$. The proof that $U(f,\mu) \subseteq U(f,\mu')$ is analogous.

First, if $V(w,\mu') = \{\emptyset\}$, then there is nothing to prove. Suppose that $V(w,\mu') \neq \{\emptyset\}$, and let $f \in V(w,\mu')$. Then, $w \in Ch(\mu'(f) \cup w, P(f))$. Strong substitutability implies then $w \in Ch(\mu(f) \cup w, P(f))$, as $\mu'(f)\,R(f)\,\mu(f)$ because $\mu \leq \mu'$. □

Let $\mathscr{V}' = \{v \in \mathscr{V} : v(s)\,R(s)\,\emptyset$ for all $s \in F \cup W\}$. We need to work on the set $\mathscr{V}'$ instead of $\mathscr{V}$ because v_0 and v_1 are the smallest and largest, respectively, elements of $\mathscr{V}'$. Note that $T(\mathscr{V}) \subseteq \mathscr{V}'$, so there is no loss in working with $\mathscr{V}'$.

LEMMA 14.3. *If P is substitutable, then $T|_{\mathscr{V}'}$ is increasing when $\mathscr{V}'$ is endowed with $\leq^B$. If P is strongly substitutable, then $T|_{\mathscr{V}'}$ is increasing when $\mathscr{V}'$ is endowed with $\leq$.*

PROOF. We show that $T|_{\mathscr{V}'}$ is increasing when $\mathscr{V}'$ is endowed with order $\leq^B$. That is, whenever $\mu \leq^B \mu'$ we have $(T\mu) \leq^B (T\mu')$. The proof for $\leq$ follows along the same lines, using Lemma 14.2 instead of 14.1.

Let $\mu \leq^B \mu'$ and fix $f \in F$ and $w \in W$. Lemma 14.1 says that $U(f,\mu) \subseteq U(f,\mu')$. We first show that

$$Ch(U(f,\mu'),P(f)) = Ch([Ch(U(f,\mu'),P(f)) \cup Ch(U(f,\mu),P(f))],P(f)). \tag{39}$$

To see this, let $S \subseteq Ch(U(f,\mu'),P(f)) \cup Ch(U(f,\mu),P(f))$. Then $S \subseteq U(f,\mu) \cup U(f,\mu') = U(f,\mu')$, so $Ch(U(f,\mu'),P(f))\ R(f)\ S$. But $Ch(U(f,\mu'),P(f)) \subseteq Ch(U(f,\mu'),P(f)) \cup Ch(U(f,\mu),P(f))$, so we have established (39).

Now, $(T\mu')(f) = Ch(U(f,\mu'),P(f))$ and $(T\mu)(f) = Ch(U(f,\mu),P(f))$, so (39) implies that

$$(T\mu')(f) = Ch([(T\mu')(f) \cup (T\mu)(f)],P(f))).$$

The proof for $(T\mu')(w)$ is analogous. □

Now $T|_{\mathscr{V}'} : \mathscr{V}' \to \mathscr{V}'$ is increasing and $\mathscr{V}'$ is a lattice (Remark 9.1). We have $T(\mathscr{V}) \subseteq \mathscr{V}'$ so $\mathscr{E}(P) \subseteq \mathscr{V}'$, and $\mathscr{E}(P)$ equals the set of fixed points of $T|_{\mathscr{V}'}$. So Tarski's fixed point theorem implies that $\langle \mathscr{E}(P), \leq^B \rangle$ and $\langle \mathscr{E}(P), \leq \rangle$ are nonempty lattices. Item (ii) in Theorem 9.7 follows from standard results (Topkis 1998, Chapter 4).

This finishes the proof of Theorem 9.7.

14.2 *Proof of Theorem 9.8*

We first prove item (i).

Let ν, $\nu' \in \mathscr{E}(P)$ be such that $\nu'(w)\ R(w)\ \nu(w)$ for all $w \in W$. Suppose, by way of contradiction, that there is some $\overline{f} \in F$ such that $\nu'(\overline{f})\ P(\overline{f})\ \nu(\overline{f})$. Let $C = Ch(\nu(\overline{f}) \cup \nu'(\overline{f}), P(\overline{f}))$, so $C\ R(\overline{f})\ \nu'(\overline{f})\ P(\overline{f})\ \nu(\overline{f})$. But $\nu \in \mathscr{E}(P)$ implies that $\nu(\overline{f}) = Ch(\nu(\overline{f}), P(\overline{f}))$ (Lemma 11.4), so $C \nsubseteq \nu(\overline{f})$. Hence there is $\overline{w} \in C \setminus \nu(\overline{f})$; note that $w \in \nu'(f)$. Now

$$\overline{w} \in C = Ch(\nu(\overline{f}) \cup \nu'(\overline{f}) \cup \overline{w}, P(\overline{f}))$$

and the substitutability of $P(\overline{f})$ imply that $\overline{w} \in Ch(\nu(\overline{f}) \cup \overline{w}, P(\overline{f}))$. So $\overline{f} \in V(\overline{w}, \nu)$.

Now, $\overline{w} \in \nu'(\overline{f}) \setminus \nu(\overline{f})$ implies $\overline{f} \in \nu'(\overline{w}) \setminus \nu(\overline{w})$. Then $\nu'(\overline{w})\ R(\overline{w})\ \nu(\overline{w})$ implies that $\nu'(\overline{w})\ P(\overline{w})\ \nu(\overline{w})$, as $P(\overline{w})$ is strict. But $\nu'(\overline{w}) = Ch(\nu'(\overline{w}), P(\overline{w})) = Ch(\nu'(\overline{w}) \cup \overline{f}, P(\overline{w}))$ by Lemma 11.4. So strong substitutability implies that $\overline{f} \in Ch(\nu(\overline{w}) \cup \overline{f}, P(\overline{w}))$. Since $\overline{f} \notin \nu(\overline{w})$, we obtain $\nu(\overline{w}) \cup \overline{f}\ P(\overline{w})\ \nu(\overline{w})$. This contradicts $\nu \in \mathscr{E}(P)$, since we showed $\overline{f} \in V(\overline{w}, \nu)$ and $\nu \in \mathscr{E}(P)$ imply $\nu(\overline{w}) = Ch(V(\overline{w}, \nu), P(\overline{w}))$.

To prove item (ii) in the theorem, note that when $P(F)$ is strongly substitutable the model is symmetric, and the argument above holds with firms in place of workers, and workers in place of firms. □

14.3 *Proof of Theorem 9.11*

We first prove that $\langle \mathscr{E}(P), \leq \rangle$ is a sublattice of $\langle \mathscr{V}, \leq \rangle$. That $\langle \mathscr{E}(P), \leq \rangle$ is distributive follows then immediately. We need to verify that the lattice operations $\vee$ and $\wedge$ in $\mathscr{V}$ are the lattice operations in $\langle \mathscr{E}(P), \leq \rangle$.

Let $v^1, v^2 \in \mathscr{E}(P)$. Let $v = v^1 \vee v^2$ in $\mathscr{V}$. We prove that v is the join of v^1, v^2 in $\langle \mathscr{E}(P), \leq \rangle$. The proof for $v^1 \wedge v^2$ is analogous.

By hypothesis v is a matching; so

$$w \in v(f) \rightarrow f \in v(w).$$

We prove that $v \in \mathscr{E}(P)$. Suppose, by way of contradiction, that there is $\overline{f}$ such that $(Tv)(\overline{f}) \neq v(\overline{f})$. Without loss of generality, say that $v(\overline{f}) = v^1(\overline{f})\ R(\overline{f})\ v^2(\overline{f})$. Since $v^1 \in \mathscr{E}(P)$, v^1 is individually rational (Lemma 11.4), so $\overline{f} \in Ch(v^1(w), P(w)) = Ch(v^1(w) \cup \overline{f}, P(w))$ for all $w \in v^1(\overline{f})$. For all w, on the other hand, $v^1(w)\ R(w)\ v(w)$. So strong substitutability gives $\overline{f} \in Ch(v(w) \cup \overline{f}, P(w))$ for all $w \in v^1(\overline{f})$. Thus $v^1(\overline{f}) \subseteq U(\overline{f}, v)$. Since $(Tv)(\overline{f}) = Ch(U(\overline{f}, v), P(\overline{f}))$ and v^1 is individually rational, $(Tv)(\overline{f}) \setminus v(\overline{f}) \neq \emptyset$.

Let $\overline{w} \in (Tv)(\overline{f}) \setminus v(\overline{f})$. By substitutability, $\overline{w} \in Ch(v^1(\overline{f}) \cup \overline{w}, P(\overline{f}))$. Strong substitutability and $v^1(\overline{f})\ R(\overline{f})\ v^2(\overline{f})$ then imply $\overline{w} \in Ch(v^2(\overline{f}) \cup \overline{w}, P(\overline{f}))$. So

$$\overline{f} \in V(\overline{w}, v^i) \tag{40}$$

for $i = 1, 2$.

On the other hand $\overline{w} \in (Tv)(\overline{f})$ implies $\overline{w} \in U(\overline{f}, v)$, so

$$\overline{f} \in Ch(v(\overline{w}) \cup \overline{f}, P(\overline{w})). \tag{41}$$

Let i be such that $v(\overline{w}) = v^i(\overline{w})$. Then (40) and $v^i \in \mathscr{E}(P)$ imply $v^i(\overline{w}) \cup \overline{f} \in V(\overline{w}, v^i)$.

But we assumed $\overline{w} \notin v(\overline{f})$, so $\overline{f} \notin v^i(\overline{w})$, as v is a matching. Then $v^i(\overline{w}) \cup \overline{f} \neq v^i(\overline{w})$, which contradicts $v^i \in \mathscr{E}(P)$, given (41) and $v(\overline{w}) \cup \overline{f} \in V(\overline{w}, v^i)$.

For the rest of the theorem, we need a lemma.

Lemma 14.4. *Let P be strongly substitutable. For all f and w, for any v and v' in $\mathscr{V}$, $U(f, v \vee v') = U(f, v) \cup U(f, v')$, $U(f, v \wedge v') = U(f, v) \cap U(f, v')$, $V(w, v \vee v') = V(w, v) \cap V(w, v')$, and $V(w, v \wedge v') = V(w, v) \cup V(w, v')$.*

Proof. We prove only that $U(f, v \vee v') = U(f, v) \cup U(f, v')$ and that $V(w, v \vee v') = V(w, v) \cap V(w, v')$. The proof of the other statements is symmetric.

We first prove that $U(f, v \vee v') \subseteq U(f, v) \cup U(f, v')$. Let $w \in U(f, v \vee v')$, so $f \in Ch((v \vee v')(w) \cup f, P(w))$. Now, $(v \vee v')(f)$ equals either $v(f)$ or $v'(f)$. If $(v \vee v')(w) = v(w)$, then $f \in Ch(v(w) \cup f, P(w))$; so $w \in U(f, v)$. Similarly, if $(v \vee v')(w) = v'(w)$, then $w \in U(f, v')$. This proves that $U(f, v \vee v') \subseteq U(f, v) \cup U(f, v')$.

Second, we prove that $U(f, v) \cup U(f, v') \subseteq U(f, v \vee v')$. Let $w \in U(f, v)$, so $f \in Ch(v(w) \cup f, P(w))$. Now $v(w)\ R(w)\ (v \vee v')(w)$, so strong substitutability implies $f \in Ch((v \vee v')(w) \cup f, P(w))$. Hence $w \in U(f, v \vee v')$. This proves that $U(f, v) \cup U(f, v') \subseteq U(f, v \vee v')$. So, $U(f, v \vee v') = U(f, v) \cup U(f, v')$.

We now prove that $V(w, v \vee v') = V(w, v) \cap V(w, v')$. First we prove $V(w, v \vee v') \subseteq V(w, v) \cap V(w, v')$. Let $f \in V(w, v \vee v')$, so

$$w \in Ch((v \vee v')(f) \cup w, P(f)). \tag{42}$$

Without loss of generality, say $(v \vee v')(f) = v(f)\ R(f)\ v'(f)$. Then $(v \vee v')(f) = v(f)$ implies that $f \in V(w, v)$. Statement (42) and strong substitutability imply that $w \in Ch(v'(f) \cup w, P(f))$, as $(v \vee v')(f)\ R(f)\ v'(f)$. Thus $f \in V(w, v)$, and we obtain $V(w, v \vee v') \subseteq V(w, v) \cap V(w, v')$.

Finally, we prove that $V(w, v) \cap V(w, v') \subseteq V(w, v \vee v')$. Let $f \in V(w, v) \cap V(w, v')$, so $w \in Ch(v(f) \cup w, P(f))$ and $w \in Ch(v'(f) \cup w, P(f))$. Now, $(v \vee v')(w)$ equals either $v(w)$ or $v'(w)$, so either way $w \in Ch((v \vee v')(f) \cup w, P(f))$. Hence $f \in V(w, v \vee v')$. □

Lemma 14.4 implies immediately that ψ is a lattice homomorphism: Let $v', v \in \mathscr{V}$. For any f and w,

$$(\psi(v \vee v'))(f) = U(f, v \vee v') = U(f, v) \cup U(f, v') = (\psi v)(f) \cup (\psi v')(f)$$
$$(\psi(v \vee v'))(w) = V(w, v \vee v') = V(w, v) \cap V(w, v') = (\psi v)(f) \cap (\psi v')(f).$$

So $\psi(v \vee v') = \psi v \sqcup \psi v'$. That $\psi(v \wedge v') = \psi v \sqcap \psi v'$ is also trivial from Lemma 14.4.

We now show that $\psi|_{\mathscr{E}(P)}$ is an isomorphism onto its range. Let v, $v' \in \mathscr{E}(P)$. Let $\psi v = \psi v'$. Then for all f, $U(f, v) = U(f, v')$ so $(Tv)(f) = (Tv')(f)$. Similarly $(Tv)(w) = (Tv')(w)$ for all w. So $Tv = Tv'$. Then v, $v' \in \mathscr{E}(P)$ implies $v = v'$, as $v = Tv$ and $v' = Tv'$. Hence ψ is one-to-one, as $\psi v = \psi v'$ implies $v = v'$.

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