Supplement to "Social distance and network structures"

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Appendix B: Alternative microfoundation of the kth norm and cutoff rule

In the main section, we have postulated that the benefit of link ij is decreasing in the kth norm distance between i's and j's types, and this formulation led to cutoff rules under linear cost functions. Here we provide an alternative microfoundation of the cutoff rule model by considering a model where connected agents are involved with strategic interactions across m dimensions. The kth norm and the value of k are endogenously derived in this model, which also facilitates the interpretation of the comparative-statics results in k.

Suppose that, if i and j are connected, each receives the benefit from the link

$$\sum_{h=1}^{m} v\big(|x_{ih} - x_{jh}|\big),$$

where, for each $h = 1, 2, \dots, m$,

$$v\big(|x_{ih} - x_{jh}|\big) = \begin{cases} \bar{v} & \text{if } |x_{ih} - x_{jh}| \le \hat{d}, \\ 0 & \text{if } |x_{ih} - x_{jh}| > \hat{d} \end{cases}$$

for some $\hat{d} > 0$ and $\bar{v} > 0$. We interpret $v(|x_{ih} - x_{jh}|)$ to be the payoff obtained at dimension h. As shown by the examples at the end of this section, this type of benefit functions naturally arises in various situations.

The payoff of agent i at network g is thus given by

$$u_i(g) = \left(\sum_{j \in N_i(g)} \sum_{h=1}^{m} v\big(|x_{ih} - x_{jh}|\big) \right) - \bar{c} q_i, \tag{A1}$$

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where $\bar{c} > 0$. It is straightforward to see that, for any realization of types $(x_1, \dots, x_n)$, there is a unique pairwise stable network that is generated by a cutoff rule under the kth norm with $k = \lceil \frac{\bar{c}}{\bar{v}} \rceil$, where $\hat{d}$ is the corresponding cutoff value.[1] Conversely, for any k and $\hat{d}$, we can find a payoff function of the form (A1) such that, for any realization of types $(x_1, \dots, x_n)$, a unique network formed by the cutoff rule under the kth norm and cutoff $\hat{d}$ is a unique pairwise stable network.

EXAMPLE 1 (Repeated Prisoner's Dilemma with Imperfect Monitoring). Consider the situation in which, at each dimension $h = 1, \dots, m$, each of connected agents i and j play an infinitely repeated prisoner's dilemma with imperfect public monitoring in discrete time $t = 1, 2, \dots$ with discount factor $\beta < 1$. Agents receive the following payoffs at each period (but do not observe them over the course of play), where $T, T' > 0$ and $\bar{V} > T - T'$:

	C	D
C	$\bar{V}, \bar{V}$	$-T', \bar{V} + T$
D	$\bar{V} + T, -T'$	$0, 0$

There are two public signals G (good) and B (bad). Action profile (C, C) always results in G, but at (C, D), (D, C), or (D, D), B occurs with probability $q(|x_{ih} - x_{jh}|)$ that is continuous and strictly decreasing in the type difference with $q(1) = 0$: monitoring is less precise if agents are farther away from each other. We restrict attention to the strategies such that the action at a dimension depends only on the past signals at that dimension, and we consider perfect public equilibria that maximize the sum of the two agents' discounted sums of payoffs. Under this assumption, it is without loss to focus on the following two types of strategies: (i) the strategy where each agent chooses C if and only if only signal G has been observed in the past, and (ii) the unconditional repetition of D. Let $\bar{v} := \frac{\bar{V}}{1-\beta}$. Then, for sufficiently high $\beta < 1$, there exists $\hat{d} = q^{-1}(\frac{T}{\beta \bar{v}}) \in (0, 1)$ such that, at each dimension, each agent receives the equilibrium payoff $\bar{v}$ if $|x_{ih} - x_{jh}| \le \hat{d}$, but the unique equilibrium is the repetition of D and thus agents receive 0 if $|x_{ih} - x_{jh}| > \hat{d}$.[2] $\diamondsuit$

EXAMPLE 2 (Repeated Prisoner's Dilemma with Perfect Monitoring). Consider the following game where each of connected agents chooses either to cooperate (C) or not (D) at each dimension $h = 1, 2, \dots, m$, where $\bar{V}, T', \psi(\cdot) > 0$, and $\bar{V} > \psi(1) - T'$:

[1] There is an alternative but equivalent formulation in which the benefit from a neighbor is constant at $\bar{v} > 0$, while the agents incur a cost at each dimension, as in

$$u_i(g) = \bar{v} q_i - \left(\sum_{j \in N_i(g)} \sum_{h=1}^{m} \psi\big(|x_{ih} - x_{jh}|\big) \right),$$

where $\psi(|x_{ih} - x_{jh}|)$ is equal to 0 if $|x_{ih} - x_{jh}| \le \hat{d}$ and is equal to $\bar{c} > 0$ if $|x_{ih} - x_{jh}| > \hat{d}$. This leads to the cutoff rule using the kth norm with $k = \lceil \frac{\bar{v}}{\bar{c}} \rceil$.

[2] Note that (C, D) or (D, C) cannot be enforced at any history in this game because an agent would have a strict incentive to choose D at such a history.

	C	D
C	$\bar{V}, \bar{V}$	$-T', \bar{V} + \psi(\lvert x_{ih} - x_{jh}\rvert)$
D	$\bar{V} + \psi(\lvert x_{ih} - x_{jh}\rvert), -T'$	$0, 0$

We assume that $\psi(\cdot)$ is continuous and strictly increasing: the temptation to defect is higher if agents are farther away from each other. Agents play the infinitely repeated game with perfect monitoring with discount factor $\beta < 1$. Again, we restrict attention to the strategies such that actions at each dimension $h = 1, 2, \ldots, m$ at each period do not depend on the past actions at other dimensions, and we suppose that agents play a subgame-perfect equilibrium that maximizes the sum of the two agents' discounted sums of payoffs. Again, without loss, we focus only on the grim-trigger strategy. Namely, agents start by playing C, and play D if and only if the history contains at least one D by any agent. Let $\bar{v} := \frac{\bar{V}}{1-\beta}$. If $\frac{\beta\bar{V}}{1-\beta} \in [\psi(0), \psi(1)]$, then the following statements are true with the unique $\hat{d} := \psi^{-1}(\frac{\beta\bar{V}}{1-\beta})$: If $|x_{ih} - x_{jh}| \leq \hat{d}$, then the grim-trigger strategy profile is sustained as a subgame-perfect equilibrium, and agents receive the equilibrium payoff $\bar{v}$. If, however, $|x_{ih} - x_{jh}| > \hat{d}$ holds, then the unique subgame-perfect equilibrium is for agents to always choose D, and agents receive payoff 0. $\Diamond$

Example 3 (Coordination Game). Consider the following static game where each of connected agents chooses either A or B at each dimension $h = 1, 2, \ldots, m$,

	A	B
A	$\bar{v}, \bar{v}$	$-\phi(\lvert x_{ih} - x_{jh}\rvert), 0$
B	$0, -\phi(\lvert x_{ih} - x_{jh}\rvert)$	$0, 0$

where $\phi(\cdot) > 0$ represents the cost of miscoordination, which is strictly increasing: miscoordination is more costly if agents are farther away from each other.

Pick $\bar{v} > 0$ and $\hat{d} > 0$ such that $\phi(\hat{d}) = \bar{v}$. This game has multiple strict Nash equilibria, (A, A) and (B, B). We assume that agents play a strict risk-dominant equilibrium, which generically exists.[3] This implies that (A, A) is played if $|x_{ih} - x_{jh}| < \hat{d}$, and (B, B) is played if $|x_{ih} - x_{jh}| > \hat{d}$. $\Diamond$

Appendix C: Omitted proofs for the main sections

C.1 *Proof of Lemma 1*

Let $c(q) = c_1 q$ be the linear cost function, where $c_1 > 0$.

Part (i): (a) Existence of a Pairwise Stable Network. Consider the maximum of ds that satisfies $b(d) - c_1 \geq 0$, and denote it by $\hat{d}$ (the maximum exists because b is nonincreasing

[3] There are various justifications behind this selection criterion, such as global games (Carlsson and Van Damme 1993), information robustness (Kajii and Morris 1997), and evolutionary dynamics (Kandori et al. 1993, Young 1993), among others.

and continuous from the left). We have

$$\Delta c(q) = \big(c_1(q+1)\big) - (c_1 q) = c_1 \quad \text{for all } q.$$

Network g is pairwise stable if and only if (α) there is no link $ij \in g$ such that $u_i(g) < u_i(g - ij)$ and (β) there is no link $ij \notin g$ such that $u_i(g) \le u_i(g + ij)$. Now, since $\Delta c(q) = c_1$ for all q, (α) is equivalent to saying that there is no $ij \in g$ such that $0 > b(d(i,j)) - c_1$, and (β) is equivalent to saying that there is no $ij \notin g$ such that $0 \le b(d(i,j)) - c_1$. Noting that $b(d(i,j)) - c_1 \ge 0 \Longleftrightarrow d(i,j) \le \hat{d}$, we have that $g = \{ij : d(i,j) \le \hat{d}\}$ is pairwise stable. Thus, a pairwise stable network exists.

(b) Uniqueness of the Pairwise Stable Network. Suppose that there are two distinct pairwise stable networks, g and g'. Without loss of generality, there exists a pair of agents $i, j \in N$ such that $ij \in g$ and $ij \notin g'$. But $ij \in g$ and (α) in part (i)(a) of this proof imply $b(d(i,j)) - c_1 \ge 0$, while $ij \notin g'$ and (β) in part (i)(a) of this proof imply $b(d(i,j)) - c_1 < 0$. Contradiction.

(c) Efficiency of the Pairwise Stable Network. Suppose, to the contrary, that the pairwise stable network g is not efficient. That is, suppose that there is another network g' in which the sum of utilities of all the agents is strictly larger in g' than in g. Let $L_1 = g \setminus g'$ and $L_2 = g' \setminus g$. That is, g' is obtained from g by deleting all the links in L_1 and adding all the links in L_2. Note that the order of deletion and addition of links does not matter for the efficiency from the resulting networks by the definition of efficient networks. Now, for all $ij \in L_1$, we have $b(d(i,j)) - c_1 \ge 0$ from part (i)(a) of this proof, so the sum of utilities strictly decreases by deletion of links in L_1 unless L_1 consists only of links ij such that $d(i,j) = c_1$. Next, for all $ij \in L_2$, we have $b(d(i,j)) - c_1 < 0$ from part (i)(a) of this proof, so the sum of utilities strictly decreases by addition of links in L_2 if L_2 is not empty, and stays constant if it is empty. Hence, the only way that g' can be efficient is that L_1's only elements are the links ij such that $d(i,j) = c_1$, and L_2 is empty. But as deleting the links ij such that $d(i,j) = c_1$ does not change the utility of either i or j, and hence, it does not change the sum of utilities, g' has the same sum of utilities as g. But this contradicts our starting assumption that g' is such that the sum of utilities of all the agents is strictly larger in g' than in g. This completes the proof.

(d) Existence of a Homogeneous Cutoff Value Profile. In parts (i)(a) and (i)(b) of this proof, we have shown that the unique pairwise stable network is $g = \{ij : d(i,j) \le \hat{d}\}$. Let a cutoff value profile be such that $\hat{d}_i = \hat{d}$ for all $i \in N$. This cutoff value profile is homogeneous by definition, and clearly generates network g.

Part (ii): Existence of pair (b, c). Fix a network g that is generated by a cutoff rule with a homogeneous cutoff value profile. It suffices to provide one example of (b, c) pair such that g is pairwise stable with respect to the pair (b, c). Uniqueness and efficiency follow directly from parts (i)(b) and (i)(c), respectively.

Let the homogeneous cutoff value be $\hat{d}$. Consider a pair of functions $b(d) = a \cdot \frac{\hat{d}}{d}$ and $c(q) = a \cdot q$ for some $a > 0$. These functions satisfy the assumptions made in Section 3.1. Notice that the benefit from forming links when the distance is very short decreases fast if a is small, and the marginal cost of forming an additional link is large if a is large.

Now notice that $ij \in g$ implies $d(i,j) \le \hat{d}$, which implies $b(d(i,j)) - a = a \cdot \frac{\hat{d}}{d(i,j)} - a \ge 0$, which in turn implies that the marginal benefit for each of agents i and j from link ij is no less than the marginal cost. Also, $ij \notin g$ implies $d(i,j) > \hat{d}$, which implies $b(d(i,j)) - a = a \cdot \frac{\hat{d}}{d(i,j)} - a < 0$, which in turn implies that the marginal benefit for each of agents i and j from link ij is strictly less than the marginal cost. Hence g is pairwise stable. Thus the proof is complete. □

C.2 *Proof of Corollary 1*

Part (i) is straightforward from the formula in Theorem 1.

We consider part (ii). From the formula in Theorem 1,

$$\mathrm{Cl}^*(k+1,m) = \binom{m}{k+1}^{-1}\left(\frac{3}{4}\right)^{k+1} = \frac{(k+1)!(m-k-1)!}{m!}\left(\frac{3}{4}\right)^{k+1} = \mathrm{Cl}^*(k,m)\frac{3(k+1)}{4(m-k)}.$$

Taking logs, we get

$$\log\bigl(\mathrm{Cl}^*(k+1,m)\bigr) - \log\bigl(\mathrm{Cl}^*(k,m)\bigr) = \log\left(\frac{3(k+1)}{4(m-k)}\right).$$

Hence, $\mathrm{Cl}^*(k+1,m) \geq \mathrm{Cl}^*(k,m)$ is equivalent to $\frac{3(k+1)}{4(m-k)} \geq 1$, or $k \geq \frac{4}{7}m - \frac{3}{7}$, completing the proof. □

Appendix D: Omitted proofs for other sections

D.1 *Proof of Proposition 4*

Throughout this proof, we denote distance by $d(i,j)$ instead of $d(x_i,x_j)$ to lighten the notation. Fix the types of agents, $(x_1,\dots,x_n)$. We ignore the possibility that there exist $h,i,j,\in N$ such that $d(i,j) = d(i,h)$ or that there exist $i,j \in N$ and $q \in \mathbb{N}$ such that $b(d(i,j)) = \Delta c(q-1)$, because almost surely such events do not occur. This in particular implies that $N_i(g) \neq N_i(g') \Rightarrow u_i(g) \neq u_i(g')$. We consider the following algorithm that generates a unique network. We will show in the sequel that the algorithm stops in finite steps, and the generated network is pairwise stable and is generated by a cutoff rule. Moreover, we will show that the generated network is strongly stable if the cost function is concave or linear.

Algorithm. *Step* 1. Each player $i \in N(1) := N$ proposes a "request":

$$r_i(1) = \arg\max_{r'_{i1} \subseteq N(1)\setminus\{i\}} u_i\bigl(\{ij \mid j \in r'_{i1}\}\bigr).$$

Generate a network $g' := g(0) \cup \{ij \mid j \in r_i(1) \text{ and } i \in r_j(1)\} \in G(N)$, where we set $g(0) = \varnothing$. Delete $k'l' = \arg\max_{i\in N, kl\in g'}\{u_i(g'-kl) - u_i(g')\}$ if $u_i(g'-k'l') - u_i(g')$ is positive. Let $g'' = g'\setminus\{k'l'\}$. Then delete $k''l'' = \arg\max_{i\in N,kl}\{u_i(g''-kl) - u_i(g'')\}$ if $u_i(g''-k''l'') - u_i(g'')$ is positive. Continue this procedure until the generated network $\hat{g}$ satisfies the property that each link ij satisfies $u_i(\hat{g}-ij) < u_i(\hat{g})$. Let the resulting network be $g(1)$.

Step t. Each player $i \in N(t) := N(t-1)\setminus\{j : r_j(t-1) = \varnothing\}$ proposes a "request":

$$r_i(t) = \arg\max_{r'_{it} \subseteq N(t)\setminus[\{i\}\cup N_i(g(t-1))]} u_i\bigl(\{ij \mid j \in r'_{it}\} \cup g(t-1)\bigr).$$

Generate a network $g' := g(t-1) \cup \{ij | j \in r_i(t) \text{ and } i \in r_j(t)\}$. Delete $k'l' = \arg\max_{i \in N, kl \in g'} \{u_i(g' - kl) - u_i(g')\}$ if $u_i(g' - k'l') - u_i(g')$ is positive. Let $g'' = g' \setminus \{k'l'\}$. Then delete $k''l'' = \arg\max_{i \in N, kl \in g''} \{u_i(g'' - kl) - u_i(g'')\}$ if $u_i(g'' - k''l'') - u_i(g'')$ is positive. Continue this procedure until the generated network $\hat{g}$ satisfies the property that each link $ij \in \hat{g}$ satisfies $u_i(\hat{g} - ij) < u_i(\hat{g})$. Let $g(t)$ be the resulting network.

Let $\bar{t}$ be the first period, if any, such that $N(\bar{t}) = \varnothing$. If such a period does not exist, then denote $\bar{t} = \infty$.

Let us give an intuitive explanation about the algorithm. For each Step t, $N(t)$ is the set of "remaining agents." Each remaining agent makes a request to form links to some of the remaining agents, which would make him better off than the current network if it was accepted by all agents included in it. However, at each step, all the requests are not necessarily satisfied. Instead, we require that only links that are requested by both agents involved are actually formed. Hence, it is possible that some portion of a request is satisfied while the other portion is not satisfied. In such cases, it may be that, after the formation of links based on the requests, some agents have incentives to delete links that currently exist. Such links are deleted in the "deletion procedure" in each step of the algorithm. Step by step, links are gradually formed, and eventually some agents have empty requests. Such agents are removed from the algorithm, and can never be made a request or be able to make a request by themselves. Eventually, at some step, no agent remains, and the algorithm stops at such a step.

We prove the following lemmas to complete the proof of Proposition 4.

Lemma A1. *For every $t \leq \bar{t}$, if $i \in N(t)$, $k \in r_i(t)$, and $l \in N(t) \setminus r_i(t)$, then $d(i,k) < d(i,l)$.*

That is, i's request $r_i(t)$ is a set of agents who are closer to i than anyone who is in $N(t)$ but is not included in the request.

Lemma A2. *The inequality $\bar{t} < \infty$ holds, and $g(\bar{t})$ is unique.*

Hence, the algorithm stops in a finite number of steps, generating a unique network.

Lemma A3. *We have that Network $g(\bar{t})$ is pairwise stable.*

Lemma A4. *There exists $\hat{d} = (\hat{d}_1, \ldots, \hat{d}_n)$ such that $g(\bar{t})$ is generated by a cutoff rule with $\hat{d}$.*

To establish Lemma A4, we first prove the following claim.

Claim 1. *Suppose c is convex. If $j \in r_i(t)$, then $\forall t' > t$ such that $i, j \in N(t')$, either $j \in r_i(t')$ or $ij \in g(t'-1)$ holds.*

Claim 1 implies the following claim.

Claim 2. *Let $g = g(\bar{t})$, and suppose $ij \notin g$ and $d(i,j) < \max_{k \in N_i(g)}\{d(i,k)\}$. Then $u_j(g + ij) < u_j(g)$ holds.*

Claim 2 implies Claim 3, which in turn implies Lemma A4.

Claim 3. *Let $g = g(\bar{t})$, and suppose $ij \notin g$ and $d(i,j) < \max_{k \in N_i(g)}\{d(i,k)\}$. Then $d(i,j) > \max_{l \in N_j(g)}\{d(j,l)\}$ holds.*

Lemma A5. *Suppose c is linear or convex. Then $g(\bar{t})$ is strongly stable.*

Lemma A6. *Suppose c is linear or convex. Then a strongly stable network is unique.*

Proof of Lemma A1. Note that $r_i(t)$ maximizes the sum of additional benefits that i obtains minus that of additional costs that he incurs. Separability of u and the definition of r_i imply

$$r_i(t) = \arg \max_{r'_{it} \subseteq N(t) \setminus [\{i\} \cup N_i(g(t-1))]} \left[\sum_{j \in r'_{it}} b\big(d(i,j)\big) - \sum_{s=0}^{\sharp r'_{it}-1} \Delta c\big(q_i\big(g(t-1)\big) + s\big) \right].$$

Notice that the second term of the right hand side of the above equality depends only on i's degree but not on the identities of agents in r'_{it}.

Suppose, to the contrary, that there exist $i, k, l \in N(t)$ such that $d(i,k) > d(i,l)$, $k \in r_i(t)$, and $l \in N(t) \setminus r_i(t)$. Then, depriving $r_i(t)$ of k and adding l to $r_i(t)$ strictly increases i's additional benefit (the first term of the right hand side of the above equality) with i's additional cost (the second term) unchanged. This contradicts the assumption that $r_i(t)$ is the maximizer of the right hand side of the above equality. This completes the proof. □

Proof of Lemma A2. Since there is no tie in distances, for each t and each $i \in N$, $r_i(t)$ is uniquely determined. Therefore the algorithm generates a unique network, if it ends in finite steps.

Now we prove that the algorithm ends in finite steps. The algorithm can be regarded as a deterministic dynamic process over discrete time $t = 1, 2, \ldots$, defined on state space $G(N) \times 2^N$, where the state at t is $(g(t-1), N(t))$. Note that the number of states is finite.

We first show that this process is monotone. To see this, notice that the set $N(t)$ is nonincreasing. Hence it suffices to show that $g(t-1)$ is nondecreasing. To show this, we will prove that no link in $g(t-1)$ is not deleted in the "deletion procedure" at t (i) with a convex or linear cost function, and (ii) with a concave cost function.

First, consider case (i). We show that there is no agent deleting his links in the algorithm, when c is convex or linear. By the definition of the request, for each t, $i \in N(t)$, and $j \in r_i(t)$, we have

$$\begin{aligned} b\big(d(i,j)\big) &> \Delta c\big(q_i\big(g(t-1)\big) + \sharp r_i(t) - 1\big) \\ &\geq \Delta c\big(q_i\big(g(t-1)\big) + s\big), \end{aligned}$$

where $0 \leq s < \sharp r_i(t)$. This ensures that however i's requested links are actually formed, he cannot become better off by deleting his newly formed links.

Second, consider case (ii). At Step 1, the statement trivially holds, since $g(0) = \varnothing$. We have, by the construction of the algorithm, $\Delta c(q_i(g(t)) - 1) < b(d(i,j))$ for all $ij \in g(t)$.

Now consider Step $t+1$ and suppose that i becomes better off by deleting links in $g(t)$. Let ij be the first link that is deleted from $g(t)$. It must be the case that $\Delta c(q_i(g(t)) + r - 1) > b(d(i,j))$ for some $0 \le r \le \sharp r_i(t)$. But then we would have $\Delta c(q_i(g(t)) + r - 1) > \Delta c(q_i(g(t)) - 1)$, which contradicts the assumption that Δc is decreasing.

Hence, the process is monotone. Therefore, it suffices to show that there does not exist an event in which the process remains in the same state such that $N(t) \neq \varnothing$. This event could happen only if all the remaining agents make nonempty requests, and any of agents' requests are not fulfilled in the step. That is,

$$\forall i \in N(t), \quad \big[r_i(t) \neq \varnothing\big] \quad \text{and} \quad \big[\forall k \in r_i(t)\ i \notin r_k(t)\big].$$

Suppose that this is true at Step t.

The simplest case is as follows: $N(t) = \{1,2,3\}$, $r_1(t) = \{2\}$, $r_2(t) = \{3\}$, and $r_3(t) = \{1\}$. However, Lemma A1 implies that $d(1,2) < d(1,3)$, $d(2,3) < d(2,1)$, and $d(3,1) < d(3,2)$. Contradiction.

Generally, there must exist a sequence of agents $(1,2,\ldots,n')$ in $N(t)$ (with an appropriate renaming) such that $2 \in r_1(t)$, $3 \in r_2(t), \ldots, n' \in r_{n'-1}(t)$ and $1 \in r_{n'}(t)$, while $1 \notin r_2(t), 2 \notin r_3(t), \ldots, n'-1 \notin r_{n'}(t)$, and $n' \notin r_1(t)$. By Lemma A1, we have $d(1,2) < d(1,n')$, $d(2,3) < d(2,1), \ldots, d(n'-1,n') < d(n'-1,n'-2)$, and $d(n',1) < d(n',n'-1)$. Contradiction. This completes the proof. □

Proof of Lemma A3. We need to show that in $g(\bar{t})$, (i) no agent has a strict incentive to delete a link and (ii) no pair has an incentive to add a link.

To show (i), note that we have constructed the network in a way that there is no link to delete at the final step. Moreover, for agents who have left the algorithm in earlier steps, deleting their links does not increase their payoffs. This is because the set of neighbors of each agent who left earlier remains unchanged after the step at which her request was empty, and (just as in the final step,) there is no link for her to delete at that step.

To see (ii), partition the set of agents, $(P_1,\ldots,P_T)$, so that in each cell P_t of the partition, agents contained in it have empty requests at Step t. Consider an agent i in a partition P_t. At Step t, there exists no agent j in $\bigcup_{l=t}^T P_l$ such that i would be better off by connecting with j at Step t. This is because otherwise j's request would not be empty at Step t. After Step t, his degree does not change until the algorithm stops; hence, i does not have an incentive to form a link with agents in $\bigcup_{l=t}^T P_l$. Suppose that there exists agent $j' \in P_{l'}$ with $l' < t$ such that i has an incentive to form a link with j'. However, j' does not have an incentive to form a link with agents in $\bigcup_{l=l'}^T P_l$, in particular with $i \in P_t \subseteq \bigcup_{l=l'}^T P_l$. Hence, no agent has an incentive to form a link in the resulting network. □

Proof of Claim 1. It suffices to show the statement in the case of $t' = t+1$. To see this, first suppose that $ij \in g(t+1)$, given that $j \in r_i(t)$ and $i,j \in N(t+1)$. Then this implies $ij \in g(t')$ for every $t' > t$, by the monotonicity of $g(\cdot)$, which is proved in the proof of Lemma A2. Second, suppose that $j \in r_i(t+1)$, given that $j \in r_i(t)$ and $i,j \in N(t+1)$. Then, when $i,j \in N(t+2)$, we can show that either $j \in r_i(t+2)$ or $ij \in g(t+1)$ holds, by repeating exactly the same argument as in the case of $t' = t+1$, but by replacing t with $t+1$. We can repeat this argument to show that for any $t' = t+k$ with $k > 0$, the statement of the claim holds.

Now, suppose, to the contrary, that given that $j \in r_i(t)$ and $i, j \in N(t+1)$, both $j \notin r_i(t+1)$ and $ij \notin g(t)$ hold. By Lemma A1, $k \in r_i(t+1)$ implies $k \in r_i(t)$, because of $j \in r_i(t)$ and $j \notin r_i(t+1)$. That is, we have $r_i(t+1) \subsetneq r_i(t)$, where the inclusion is strict because of j.

Since the payoff function is separable, $j \in r_i(t)$ implies $\Delta c(q_i(g(t-1)) + \sharp r_i(t) - 1) < b(d(i,j))$. Also, $j \notin r_i(t+1)$, $ij \notin g(t)$, and $j \in N(t+1)$ imply $\Delta c(q_i(g(t)) + \sharp r_i(t+1)) > b(d(i,j))$. Therefore, we have $q_i(g(t-1)) + \sharp r_i(t) \leq q_i(g(t)) + \sharp r_i(t+1)$, because Δc is increasing.

At the same time, we have $N_i(g(t)) \subseteq N_i(g(t-1)) \cup r_i(t)$ by construction. Together with $r_i(t+1) \subsetneq r_i(t)$, we obtain $N_i(g(t)) \cup r_i(t+1) \subsetneq N_i(g(t-1)) \cup r_i(t)$. This implies that we have $q_i(g(t)) + \sharp r_i(t+1) < q_i(g(t-1)) + \sharp r_i(t)$, because $N_i(g(t)) \cap r_i(t+1) = \varnothing$. But this contradicts our earlier conclusion that $q_i(g(t-1)) + \sharp r_i(t) \leq q_i(g(t)) + \sharp r_i(t+1)$. This completes the proof. □

Proof of Claim 2. Denote $k = \arg\max_{k \in N_i(g)}\{d(i,k)\}$ and $l = \arg\max_{l \in N_j(g)}\{d(j,l)\}$.

Suppose, to the contrary, that $u_j(g+ij) > u_j(g)$ holds. But from $ij \notin g$ and the pairwise stability of g, $u_i(g) > u_i(g+ij)$ must hold. That is, we must have $b(d(i,j)) < \Delta c(q_i(g))$. At the same time, by the pairwise stability of g, we have $u_i(g) > u_i(g - ik)$. That is, $b(d(i,k)) > \Delta c(q_i(g) - 1)$ holds. When c is concave or linear, this contradicts $b(d(i,j)) < \Delta c(q_i(g))$, since $\Delta c(q)$ is nonincreasing and $b(d(i,j)) > b(d(i,k))$.

Consider the case where c is convex. By Lemma A2, $r_i(t') = \varnothing$ for some t'. Since $k \in r_i(t'')$ for some $t'' < t'$, by Lemma A1, $j \in r_i(t'')$ holds. From Claim 1, we have $j \in r_i(t)$ for any $t > t''$ such that $j \in N(t)$. This implies $j \in r_i(t')$, contradicting $r_i(t') = \varnothing$.

Therefore, for c that is either concave, convex, or linear, the statement is proved. □

Proof of Claim 3. Suppose, to the contrary, that $d(i,j) < d(j,l)$ holds.

Consider the case in which c is linear or concave. From Claim 2, $u_j(g+ij) < u_j(g)$, it holds that $b(d(i,j)) < \Delta c(q_j(g))$. Since g is pairwise stable, $u_j(g - jl) < u_j(g)$, so that $b(d(j,l)) > \Delta c(q_j(g) - 1)$ holds, where l is defined in the proof of Claim 2 (we define k in the same way as in the proof of Claim 2 also). But this implies $\Delta c(q_j(g) - 1) < \Delta c(q_j(g))$, because of $b(d(i,j)) > b(d(j,l))$. This contradicts that $\Delta c(q)$ is nonincreasing.

Consider the case of convex c. First, note that, as proved in the proof of Lemma A1, there is no agent deleting his links in the algorithm, when c is convex. Then, from $ik \in g$, at some t', $k \in r_i(t')$ holds. Similarly, by $jl \in g$, at some t'', $l \in r_j(t'')$ holds. We have $j \in r_i(t')$ and $i \in r_j(t'')$ by Lemma A1 if $j \in N(t')$ and $i \in N(t'')$. Thus, it cannot be the case that $t' = t''$, as it would imply $ij \in g$.

Consider the case of $t' < t''$. Claim 1 implies that $j \in r_i(t)$ for all $t > t'$ whenever $j \in N(t)$. But then $j \in N(t'')$ implies $j \in r_i(t'')$, which would imply $ij \in g$ as there is no "deletion procedure" in the case of convex c as we have seen already. In a perfectly symmetric manner, we cannot have $t' > t''$. Thus the proof is complete. □

Proof of Lemma A4. We claim that

$$\Big(\max_{i\in N_1(g)}\big\{d(1,i)\big\},\ \max_{i\in N_2(g)}\big\{d(2,i)\big\},\dots,\ \max_{i\in N_n(g)}\big\{d(n,i)\big\}\Big)$$

is a cutoff value profile $\hat{d}=(\hat{d}_1,\hat{d}_2,\dots,\hat{d}_n)$ generating g, where $g=g(\bar{t})$.

By the definition of the cutoff rule, it suffices to show that we do not have the case in which there exists $ij\notin g$ such that $d(i,j)\le\min\{\hat{d}_i,\hat{d}_j\}$. Suppose this holds. Then $ij\notin g$ and $d(i,j)<\max_{k\in N_i(g)}\{d(i,k)\}$ hold while $d(i,j)<\max_{l\in N_j(g)}\{d(j,l)\}$. This contradicts Claim 3, so that the existence of a cutoff value profile is proved. □

Proof of Lemma A5. As shown in Lemma 1, with a linear cost function, the pairwise stable network is unique, and hence, Lemma A3 implies that the generated network is the network constructed in the proof of Lemma 1. Due to the separability of the payoff function, it is straightforward to see that the network is also strongly stable. Hence, we constrain our attention to the case in which c is convex: We prove that $g=g(\bar{t})$ is strongly stable when c is convex. Take g' that is obtainable from g via deviations by a set of agents $S\subseteq N$. The statement of the lemma is true if

$$\big[\exists s\in S\ u_s(g')>u_s(g)\big]\quad\Longrightarrow\quad\big[\exists s'\in S,u'_s(g')<u'_s(g)\big].$$

Hence, it suffices to show that it cannot be the case that $u_s(g')>u_s(g)$ for every $s\in S$. Define $D(s)=\{j\in N|sj\in g,sj\notin g'\}$ and $A(s)=\{j\in N|sj\notin g,sj\in g'\}$, that is, $D(s)$ (resp. $A(s)$) is a set of agents whose link to $s\in S$ is deleted (resp. added) in the deviations.

We are going to show that, for the profitable deviations by S to be possible, there must exist an infinite sequence of agents, denoted by $s_1,s_2,s_3,\dots\in S$, such that $s_{l+1}\in A(s_l)\backslash\{s_1,s_2,\dots,s_{l-1}\}$ for each s_l. (Since S is finite, this is impossible.) To derive this sequence, we also show that $q_{s_l}(g)\le q_{s_{l+1}}(g)$ holds and either $d(s_l,\bar{n}_l)>d(s_{l+1},\bar{n}_{l+1})$ or $q_{s_l}(g)<q_{s_{l+1}}(g)$ holds, where $\bar{n}_i$ denotes an agent whose distance to s_i is the longest among s_i's neighbors, i.e., $d(s_i,\bar{n}_i)=\max_{n\in N_{s_i}(g)}\{d(s_i,n)\}$. We prove them by the mathematical induction.

First, take an agent denoted by $s_1\in S$. Since the rule of the final step of the algorithm and the convexity of c ensures that there is no incentive to delete links, agents in S cannot be better off by only deleting their links in the deviations, implying $A(s)\neq\varnothing$ for every $s\in S$. There are two cases concerning $A(s_1)$.

- Case 1: $\forall s_i\in A(s_1)$, $u_{s_1}(g)>u_{s_1}(g+s_1s_i)$. In this case, if we have $d(s_1,s_i)>d(s_1,\bar{n}_1)$ for every $s_i\in A(s_1)$, then it would be impossible to satisfy $u_{s_1}(g)<u_{s_1}(g')$. To see this, we calculate s_1's net gain from the deviations as follows. When s_1's degree increases in the deviations, i.e., $\sharp A(s_1)>\sharp D(s_1)$, his net benefit is

$$\sum_{k\in A(s_1)}b\big(d(s_1,k)\big)-\sum_{l\in D(s_1)}b\big(d(s_1,l)\big)-\sum_{j=1}^{\sharp A(s_1)-\sharp D(s_1)}\Delta c\big(q_{s_1}(g)+j-1\big).$$

Notice that $\Delta c(q_{s_1}(g))>b(d(s_1,s_i))$ holds for all $s_i\in A(s_1)$ in this case, and that Δc is increasing. Taking any subset $\bar{A}(s_1)\subset A(s_1)$ such that $\sharp\bar{A}(s_1)=\sharp A(s_1)-\sharp D(s_1)$, the net benefit can be rearranged to

$$\left(\sum_{k\in A_1\setminus\bar{A}(s_1)} b\big(d(s_1,k)\big) - \sum_{l\in D(s_1)} b\big(d(s_1,l)\big)\right)$$
$$+\left(\sum_{k\in\bar{A}(s_1)} b\big(d(s_1,k)\big) - \sum_{j=1}^{\sharp A(s_1)-\sharp D(s_1)} \Delta c\big(q_{s_1}(g)+j-1\big)\right),$$

which is negative because $\forall s_i \in A(s_1), u_{s_1}(g) > u_{s_1}(g+s_1s_i)$. The same argument carries over to the situation where his degree does not increase in the deviations.

Hence, we can focus on the case where there exists $s_i \in A(s_1)$ such that $d(s_1,s_i) < d(s_1,\bar{n}_1)$ holds. Take such an agent s_i and denote him by s_2. The inequality $d(s_1,s_2) < d(s_1,\bar{n}_1)$ and $s_1s_2 \notin g$ imply, by Claims 2 and 3 above, $u_{s_2}(g) > u_{s_i}(g+s_1s_2)$ and $d(s_2,\bar{n}_2) < d(s_1,s_2)$.

Notice that we obtained the desired inequality $d(s_2,\bar{n}_2) < d(s_1,\bar{n}_1)$.

Now we show that $q_{s_1}(g) \le q_{s_2}(g)$: The pairwise stability of g implies $b(d(s_1,\bar{n}_1)) > \Delta c(q_{s_1}(g)-1)$, and $u_{s_2}(g) > u_{s_2}(g+s_1s_2)$ implies $b(d(s_1,s_2)) < \Delta c(q_{s_2}(g))$. By $b(d(s_1,s_2)) \ge b(d(s_1,\bar{n}_1))$, we have that $\Delta c(q_{s_1}(g)-1) < \Delta c(q_{s_2}(g))$. Hence, we also get inequality $q_{s_1}(g) \le q_{s_2}(g)$, since Δc is increasing.

- Case 2: $\exists s_i \in A(s_1), u_{s_1}(g) < u_{s_1}(g+s_1s_i)$. Take $s_i \in A(s_1)$ such that $u_{s_1}(g) < u_{s_1}(g+s_1s_i)$, and denote this agent by s_2. By the pairwise stability of g, we have $u_{s_2}(g) > u_{s_2}(g+s_0s_1)$. From $\Delta c(q_{s_1}(g)) < b(d(s_1,s_2)) < \Delta c(q_{s_2}(g))$, we get a desired inequality $q_{s_1}(g) < q_{s_2}(g)$.

Hence, we have shown the desired statements for the first step $l=1$: There exists $s_2 \in A(s_1)$ such that $q_{s_1}(g) \le q_{s_2}(g)$ holds, and either $d(s_1,\bar{n}_1) > d(s_2,\bar{n}_2)$ (Case 1) or $q_{s_1}(g) < q_{s_2}(g)$ (Case 2) holds.

Next, let us suppose that we have shown the statements up to $l=r$: There exists a sequence $(s_1,s_2,\ldots,s_r)$ in S such that $s_{l+1} \in A(s_l)\setminus\{s_1,s_2,\ldots,s_{l-1}\}$ and $q_{s_l}(g) \le q_{s_{l+1}}(g)$ holds, and either $d(s_l,\bar{n}_l) > d(s_{l+1},\bar{n}_{l+1})$ or $q_{s_l}(g) < q_{s_{l+1}}(g)$ holds for each $l=1,2,\ldots,r$.

Suppose, to the contrary, that $A(s_{r+1}) \subseteq \{s_1,s_2,\ldots,s_r\}$. We show this is impossible, for both cases below.

- Case 1′: $\forall s_i \in A(s_{r+1}), u_{s_{r+1}}(g) > u_{s_{r+1}}(g+s_{r+1}s_i)$. Due to the same discussion as in the Case 1 above, we can focus on the case where there exists $s_i \in A(s_{r+1})$ such that $d(s_{r+1},s_i) < d(s_{r+1},\bar{n}_{r+1})$. Take such an agent s_i. As we have derived $d(s_2,\bar{n}_2) < d(s_1,\bar{n}_1)$ and $q_{s_1}(g) \le q_{s_2}(g)$ in the Case 1 above, we can get $d(s_i,\bar{n}_i) < d(s_{r+1},\bar{n}_{r+1})$ and $q_{s_{r+1}}(g) \le q_{s_i}(g)$. But because $s_i \in \{s_1,s_2,\ldots,s_r\}$, at least one of them contradicts the assumed inequalities: $q_{s_l}(g) \le q_{s_{l+1}}(g)$ and either $d(s_l,\bar{n}_l) > d(s_{l+1},\bar{n}_{l+1})$ or $q_{s_l}(g) < q_{s_{l+1}}(g)$ for each $l=1,2,\ldots,r$.
- Case 2′: $\exists s_i \in A(s_{r+1}), u_{s_{r+1}}(g) < u_{s_{r+1}}(g+s_{r+1}s_i)$. Take $s_i \in A(s_{r+1})$ such that $u_{s_{r+1}}(g) < u_{s_{r+1}}(g+s_{r+1}s_i)$. By the same logic with which we obtained $q_{s_1}(g) < q_{s_2}(g)$ in the Case 2 above, we can get inequality $q_{s_{r+1}}(g) < q_{s_i}(g)$. But because

$s_i \in \{s_1, s_2, \dots, s_r\}$, this contradicts the assumed inequalities: $q_{s_l}(g) \le q_{s_{l+1}}(g)$ for each $l = 1, 2, \dots, r$.

Hence, for both cases, it must be the case that $A(s_{r+1})$ includes some $s_i \in S \backslash \{s_1, s_2, \dots, s_r\}$, such that $q_{s_{r+1}}(g) \le q_{s_i}(g)$ holds and either $d(s_{r+1}, \bar{n}_{r+1}) > d(s_i, \bar{n}_i)$ or $q_{s_{r+1}}(g) < q_{s_i}(g)$ holds. Denote this s_i by s_{r+2}.

Therefore, the mathematical induction is complete.

Since it is impossible for all the elements in the infinite sequence $s_1, s_2, \dots$ to be included in finite $S \subseteq N$, we conclude that there is no profitable deviation by any set of agents $S \subseteq N$. Hence, $g = g(\bar{t})$ is strongly stable. □

Proof of Lemma A6. Lemma 1 establishes the uniqueness for the case of linear cost functions, so we concentrate on the case of convex cost functions.

Suppose, to the contrary, that a network $g' \neq g = g(\bar{t})$ is also strongly stable. This implies that no pair of agents can profitably deviate from g'. We will show that this contradicts the finiteness of N.

Notice first that there exists some $i \in N$ such that $u_i(g) > u_i(g')$, as otherwise g would not be strongly stable. Take such an agent arbitrarily and call him agent 1. Consider two possible (exhaustive) cases.

- (i) $q_1(g) > q_1(g')$. In this case, pairwise stability of g and the convexity of the cost function imply that there exists some $i \in N_1(g) \backslash N_1(g')$ such that $u_1(g' + 1i) > u_1(g')$.
- (ii) $q_1(g) \le q_1(g')$. In this case, we can find some $i \in N_1(g) \backslash N_1(g')$ such that there exists $j \in N_1(g')$ such that $d(1, i) < d(1, j)$. Denote this j by 0. To see this, suppose, to the contrary, that for all $i \in N_1(g) \backslash N_1(g')$, for all $j \in N_1(g')$, $d(1, i) > d(1, j)$ holds. Take an arbitrary network $g'' \subset g'$ such that $N_1(g) \cap N_1(g') \subset N_1(g'')$ and $q_1(g) = q_1(g'')$. Such g'' exists because $q_1(g) \le q_1(g')$. Then we have $u_1(g) \le u_1(g'') \le u_1(g')$, where the first inequality holds because we have, when $g \neq g''$, $\forall i \in N_1(g) \backslash N_1(g'')$, $\forall j \in N_1(g'')$, $d(1, i) > d(1, j)$, and the second inequality is due to the pairwise stability of g' and the convexity of c. But this contradicts our earlier conclusion that $u_1(g) > u_1(g')$.

In either case (i) or (ii), we take such i and call him agent 2.

To complete the proof, we construct a sequence of distinct agents, $\{1, 2, \dots\}$, such that $2k \in N_{2k-1}(g) \backslash N_{2k-1}(g')$, $2k + 1 \in N_{2k}(g') \backslash N_{2k}(g)$, $2k + 2 \in N_{2k+1}(g) \backslash N_{2k+1}(g')$, and $d(2k-1, 2k) > d(2k, 2k+1) > d(2k+1, 2k+2)$ hold for each $k = 1, 2, \dots$. We considered a portion of the case with $k = 1$ in the previous paragraph. The rest of the first step can be shown to be true by following exactly the same logic as we will have below (by substituting $k = 0$), so we omit its proof.

Now, we start a mathematical induction argument to obtain the remaining parts of the infinite sequence and inequalities.

First, suppose we have shown the claims up to Step k, and consider Step $k + 1$. Then we must have $u_{2k+2}(g' + (2k + 1)(2k + 2)) < u_{2k+2}(g')$, as otherwise the pair $2k + 1$ and $2k + 2$ could profitably deviate from g' by adding $(2k + 1)(2k + 2)$ while simultaneously deleting $2k(2k+1)$. Hence, by the pairwise stability of $g \ni (2k+1)(2k+2)$ and the cost convexity, we have $q_{2k+2}(g) \leq q_{2k+2}(g')$. Notice that this implies there is an agent in $N_{2k+2}(g')$ who is not in $N_{2k+2}(g)$, because $(2k+1)(2k+2) \in g \setminus g'$. Similarly, we must have $d(2k+2, i) < d(2k+1, 2k+2)$ satisfied for all $i \in N_{2k+2}(g')$ to ensure that $2k+1$ and $2k+2$ do not profitably deviate from g'.

The two conclusions in the previous paragraph imply that we can find some $i \in N_{2k+2}(g') \backslash N_{2k+2}(g)$ such that $d(2k+2, i) < d(2k+1, 2k+2)$. If $i = 2l-1$ (resp. $2l$) for some $l = 1, 2, \ldots, k$, then we would have $d(2l-1, 2k+2) < d(2k+1, 2k+2) < d(2l-1, 2l)$ (resp. $d(2l, 2k+2) < d(2k+1, 2k+2) < d(2l-1, 2l)$) by the inductive supposition. But this contradicts Claim 3, because we have $(2l-1)2l, (2k+1)(2k+2) \in g$ and $(2l-1)(2k+2) \notin g$ (resp. $2l(2k+2) \notin g$). Hence $i \notin \{1, 2, \ldots, 2k+1\}$. Denote this i by $2k+3$.

Since we have $(2k+1)(2k+2) \in g$, $(2k+2)(2k+3) \notin g$, and $d(2k+1, 2k+2) > d(2k+2, 2k+3)$, we can apply Claim 2 to get $u_{2k+3}(g + (2k+2)(2k+3)) < u_{2k+3}(g)$. Then this implies $q_{2k+3}(g) \geq q_{2k+3}(g')$, due to the cost convexity and the pairwise stability of $g' \ni (2k+2)(2k+3)$. Again, this implies that we can find some $i \in N_{2k+3}(g) \backslash N_{2k+3}(g')$, as $(2k+2)(2k+3) \in g' \setminus g$. By applying Claim 3, we must have $d(2k+3, i) < d(2k+2, 2k+3)$. If $i \in \{2, 3, \ldots, 2k+1\}$, then we would have $d(i, 2k+3) < d(2k+2, 2k+3) < d(i, i+1) < d(i-1, i)$ by the inductive supposition. But if i is odd (resp. even), then i and $2k+3$ could profitably deviate from g' by adding $i(2k+3)$ while deleting $(i-1)i$ (resp. $i(i+1)$) and $(2k+2)(2k+3)$, respectively. Also, if $i = 1$, then the profitable deviation by 1 and $2k+3$ from g' is possible. This is because 1 would be better off by adding $(2k+3)1$ (as $(2k+3, 1) < d(2k+2, 2k+3) < d(1, 2)$) in case (i) and by adding $(2k+3)1$ and deleting 01 in case (ii), and $(2k+3)$ would be better off by adding $(2k+3)1$ and deleting $(2k+2)(2k+3)$. Hence, it must be the case that $i \notin \{1, 2, 3, \ldots, 2k+1\}$. Denoting such agent i by $2k+4$, we have shown the desired properties for Step $k+1$.

We have completed the induction. But since N is finite, it is impossible to have such an infinite sequence of distinct agents. This completes the proof. □

D.2 *Proof of Proposition 5*

Consider a point x in the type space X, and a hypothetical agent i who is situated at x, i.e., $x = x_i$.

Let $q(x_i, \delta)$ denote the number of agents in the δ-neighborhood of x_i. Then, for any $\delta > 0$ and q', $q(x_i, \delta) > q'$ holds almost surely as $n \to \infty$. Also, $\lim_{q \to \infty} \Delta c(q) = c_1 > 0$ implies that for all $\epsilon > 0$, there exists q' such that for all $q_i > q'$, $|\Delta c(q_i) - c_1| < \epsilon$.

Now take a small enough ϵ' and $\delta' > 0$ such that $b(\delta') \geq c_1 + \epsilon'$. Such ϵ' and δ' exist since $\lim_{d \to 0} b(d) > c_1$.

If i is not connected with an agent in his δ'-neighborhood, the resulting network would not be pairwise stable; hence, it is not strongly stable. Thus, i is connected with all the agents in his δ'-neighborhood. Thus, for any $\epsilon > 0$, we have $|\Delta c(q_i) - c_1| < \epsilon$ almost surely as $n \to \infty$.

Now consider links with agents outside of the δ'-neighborhood. Since strongly stability implies pairwise stability, $c_1 - \epsilon < \Delta c(q_i)$ (implied by $|\Delta c(q_i) - c_1| < \epsilon$) implies that $ij \notin g$ if $b(d(i,j)) \leq c_1 - \epsilon$ or $\hat{d} + \epsilon' \leq d(i,j)$ for $b^{-1}(c_1) = \hat{d}$ and some $\epsilon' > 0$. Also, for the same reason, $\Delta c(q_i) < c_1 + \epsilon$ (implied by $|\Delta c(q_i) - c_1| < \epsilon$) implies that $ij \in g$ if $c_1 + \epsilon \leq b(d(i,j))$ or $d(i,j) \leq \hat{d} - \epsilon''$ for the same $\hat{d}$ and for some $\epsilon'' > 0$.

Now, for any ϵ' and ϵ'', there exist agents j and k such that $\hat{d}+\epsilon' < d(i,j) < \hat{d}+2\epsilon'$ and $\hat{d}-2\epsilon'' < d(i,k) < \hat{d}-\epsilon''$ almost surely as $n \to \infty$. Also, these j and k have to satisfy $ij \notin g$ and $ik \in g$ because of the argument in the previous paragraph. Hence, agent i's cutoff value, denoted by $\hat{d}_i$, which we know exists from Proposition 4, has to satisfy $\hat{d} - 2\epsilon'' \leq \hat{d}_i < \hat{d} + 2\epsilon'$ almost surely as $n \to \infty$. Because ϵ' and ϵ'' go to zero as ϵ goes to zero by the continuity and strict decreasingness of b, and because x can be arbitrary, the proof is completed. □

D.3 *Proof of Proposition 6*

The procedure is almost the same as the proof for Theorem 1.

We only need to modify the expression in the proof of Theorem 1,

$$\frac{1}{(\hat{d})^k} \int_0^{\hat{d}} \int_0^{\hat{d}} \cdots \int_0^{\hat{d}} \frac{(2\hat{d}-y_1)(2\hat{d}-y_2)\cdots(2\hat{d}-y_k)}{(2\hat{d})^k}\, dy_1\, dy_2 \cdots dy_k,$$

to take into account the heterogeneity of the cutoff values.

The expression has a lower bound when the node in consideration has the cutoff of $\hat{d}+\epsilon$, where all the other nodes have the cutoffs $\hat{d}-\epsilon$, which is larger than

$$\begin{aligned}
&\frac{1}{(\hat{d}+\epsilon)^k} \int_{2\epsilon}^{\hat{d}+\epsilon} \int_{2\epsilon}^{\hat{d}+\epsilon} \cdots \int_{2\epsilon}^{\hat{d}+\epsilon} \frac{(2\hat{d}-y_1)(2\hat{d}-y_2)\cdots(2\hat{d}-y_k)}{(2\hat{d}+2\epsilon)^k}\, dy_1\, dy_2 \cdots dy_k \\
&= \left(\frac{\frac{3}{2}\hat{d}^2 - 2\hat{d}\epsilon - \epsilon + \frac{3}{2}\epsilon^2}{2(\hat{d}+\epsilon)^2} \right)^k .
\end{aligned}$$

Also, it has an upper bound when the node in consideration has the cutoff of $\hat{d}-\epsilon$, where all the other nodes have the cutoffs $\hat{d}+\epsilon$, which is smaller than

$$\begin{aligned}
&\frac{1}{(\hat{d}-\epsilon)^k} \int_0^{\hat{d}-\epsilon} \int_0^{\hat{d}-\epsilon} \cdots \int_0^{\hat{d}-\epsilon} \frac{(2\hat{d}-y_1)(2\hat{d}-y_2)\cdots(2\hat{d}-y_k)}{(2\hat{d}-2\epsilon)^k}\, dy_1\, dy_2 \cdots dy_k \\
&= \left(\frac{\frac{3}{2}\hat{d} + \frac{1}{2}\epsilon}{2(\hat{d}-\epsilon)} \right)^k .
\end{aligned}$$

For any $\hat{d} > 0$, both bounds converges to the same desired limit, $(\frac{3}{4})^k$ as ϵ goes to zero. This completes the proof. □

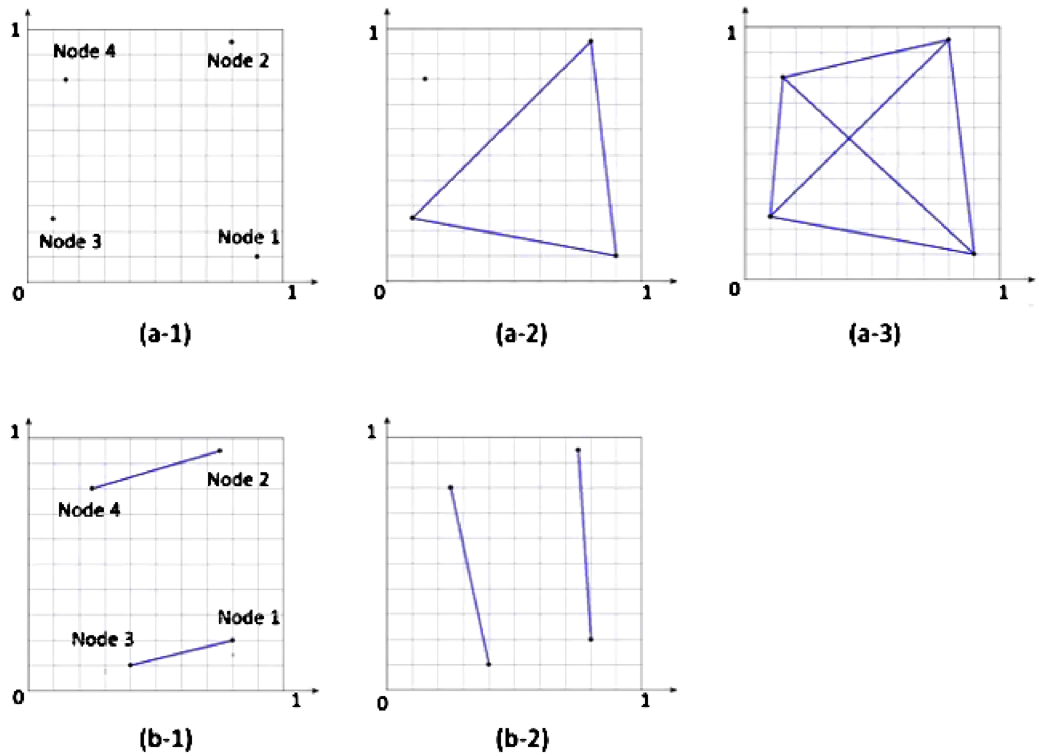

FIGURE A1. Multiple pairwise stable networks.

APPENDIX E: ADDITIONAL RESULTS

E.1 *Examples for Section 5*

With a nonlinear cost function, pairwise stability may not determine a unique network structure. Moreover, a pairwise stable network is not necessarily generated by a cutoff rule, as is illustrated by the following examples in Figure A1.

First, consider the composition of nodes in (a-1)–(a-3). There are four nodes, 1, 2, 3, and 4, located in the type space $X = [0, 1]^2$, with $x_1 = (0.9, 0.1)$, $x_2 = (0.8, 0.95)$, $x_3 = (0.1, 0.25)$, and $x_4 = (0.15, 0.8)$. We consider the case with $k = m = 2$. Calculating the distances, we get $d(x_1, x_2) = 0.85$, $d(x_1, x_3) = 0.8$, $d(x_1, x_4) = 0.75$, $d(x_2, x_3) = 0.7$, $d(x_2, x_4) = 0.65$, and $d(x_3, x_4) = 0.55$. Suppose that $b(d) = \frac{1}{d}$, $c(0) = 0$, $c(1) = 2$, $c(2) = 2.2$, and $c(3) = 2.3$. Notice that the cost function c is concave. In this case, there are three types of pairwise stable network structures, depicted in (a-1), (a-2), and (a-3), respectively. The network in (a-1) is pairwise stable because the cost to form the first link, i.e., $\Delta c(0)$, is so high that no one wants to form a link. The network in (a-2) is pairwise stable because, again, the cost for the node 4 to form the first link is so high that he does not want to form a link even though each of the other three nodes has an incentive to form a link with him. There are three other networks of this type, in each of which one agent has degree 0 and other three agents have degree 2. The network in (a-3) is also pairwise stable because the fact that the marginal cost of forming a third link, $\Delta c(2)$, is very low implies that the marginal benefit of deleting a third link is negative.

Next, in (b-1) and (b-2), we have four nodes, 1, 2, 3, and 4, located in the type space $X = [0,1]^2$, with $x_1 = (0.8, 0.2)$, $x_2 = (0.75, 0.95)$, $x_3 = (0.4, 0.1)$, and $x_4 = (0.25, 0.8)$. Again, we consider the case with $k = m = 2$. Suppose that $b(d) = \frac{1}{d}$, $c(0) = 0$, $c(1) = 1$, $c(2) = 10$, and $c(3) = 30$. Notice that c is convex. Distances between nodes are $d(x_1, x_2) = 0.75$, $d(x_1, x_3) = 0.4$, $d(x_1, x_4) = 0.6$, $d(x_2, x_3) = 0.85$, $d(x_2, x_4) = 0.5$, and $d(x_3, x_4) = 0.7$. In this case, there are at least two pairwise stable networks, depicted in (b-1) and (b-2), respectively.[4] Both networks in (b-1) and in (b-2) are pairwise stable because the marginal cost for these nodes to have a second link is very high. But the network in (b-2) is not generated by a cutoff rule. For, if it were, the cutoff value of node 1 has to be no less than 0.75 because it is connected to node 2 and $d(x_1, x_2) = 0.75$. The cutoff value of node 3 has to be also no less than 0.7 because it is connected to node 4 and $d(x_3, x_4) = 0.7$. But then $d(x_1, x_3) = 0.4 < 0.7$ implies that it has to be the case that the link 13 is formed; a contradiction.

Although we have multiplicity of pairwise stable networks in both concave and convex cost functions, the reasons for the multiplicity are quite different. Precisely, in the case of convex cost functions, it is impossible that two networks $g, g' \in G(N)$ are both pairwise stable *and* $g \subsetneq g'$, while it is possible in the case of concave cost functions, as shown in the example in Figure A1(a).

A pairwise stable network is not necessarily generated by a cutoff rule if it is not strongly stable. In the example in Figure A1, for instance, the network in (b-2) is pairwise stable, but is not (uniquely) strongly stable. So the fact that it is not generated by a cutoff rule is still consistent with the result in Proposition 4. But it is always the case that there exists a pairwise stable network that is generated by a cutoff rule. Moreover, using the notion of strong stability, we can select a smaller set (or even a singleton set under certain circumstances) of networks in which players form links as if they are using some cutoff values. Note that, as opposed to the case of linear cost functions, the cutoff value profile, if any, in a pairwise stable network under a nonlinear cost function is not necessarily homogeneous. An example is the network in Figure A1(a-2), where agents 1–3 and agent 4 cannot have a homogeneous cutoff value profile. Note that this network is not strongly stable, as the network in Figure A1(a-3) is obtainable from the network in Figure A1(a-2) via deviations by $S = \{1, 2, 3, 4\}$ and that all the agents would be better off after such deviations.

A homogeneous cutoff value profile may not exist even in strongly stable networks. Consider the composition of nodes in Figure A2. There are four nodes, 1, 2, 3, and 4, located in the type space $X = [0,1]^2$, with $x_1 = (0.7, 0.1)$, $x_2 = (0.8, 0.95)$, $x_3 = (0.2, 0.05)$, and $x_4 = (0.15, 0.4)$. We consider the case with $k = m = 2$. Suppose that $b(d) = \frac{1}{d}$, $c(0) = 0$, $c(1) = 1$, $c(2) = 5$, and $c(3) = 10$. Distances between nodes are $d(x_1, x_2) = 0.85$, $d(x_1, x_3) = 0.5$, $d(x_1, x_4) = 0.55$, $d(x_2, x_3) = 0.9$, $d(x_2, x_4) = 0.65$, and $d(x_3, x_4) = 0.35$. It is straightforward to see that there is a unique pairwise stable network, namely $g = \{12, 34\}$, as in the figure. This is also strongly stable.

[4] A network $\{14, 23\}$ is also pairwise stable.

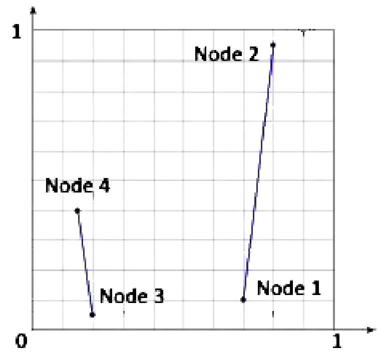

FIGURE A2. Heterogeneous cutoff values.

Now, because node 1 is connected with node 2, his cutoff value, if any, has to be no less than 0.85. But because node 3 is not connected with node 1, his cutoff value, if any, has to be strictly less than 0.5. This implies that we cannot find any homogeneous cutoff value profile. Hence, this example shows that even in a strongly stable network, a homogeneous cutoff value profile may not exist. However, as Proposition 4 shows, a heterogeneous cutoff value profile must exist. For example, $(\hat{d}_1, \hat{d}_2, \hat{d}_3, \hat{d}_4) = (0.85, 0.85, 0.35, 0.35)$ serves as a heterogeneous cutoff value profile.

E.2 *Robustness of the main results against non-uniform type distribution*

We examine the extent to which the comparative statics provided in Corollaries 1 and 2 go through even under non-uniform type distributions. As is clear from the intuition explained above, the clustering coefficient is higher if there is more asymmetry in the size of various types of neighbors of a given agent. This means that the comparative-statics results are likely to be robust if the type distribution is not too asymmetric. To make this intuition precise, we formalize a measure of asymmetry of type distributions and derive a bound such that if the level of asymmetry is below that bound, our comparative-statics results go through.[5]

Given a distribution f on the m-dimensional space X, define the measure of asymmetry of f[6] as

$$a^f = \lim_{d \to 0} \max_{w \in [0,1]^m} \max_{S,T \subseteq \{1,\dots,m\}, |S|=|T|=k} \frac{t^f(w,d,T)}{t^f(w,d,S)},$$

where

$$t^f(w,d,S) = \frac{\text{Prob}_{y \sim f}\big(|y_j - w_j| \le d \text{ for all } j \in S\big)}{\text{Prob}_{y \sim f}\big(|y_j - w_j| \text{ for all } j \in T \text{ such that } T \subseteq \{1,\dots,m\} \text{ and } |T| \ge k\big)}$$

[5]In the main model, agents are connected if there are at least k dimensions on which their types are within cutoff $\hat{d}$. More generally we could allow the cutoff to be different across dimensions. One can interpret the results in this section as providing conditions under which the comparative statics of the main model is applicable to such a more general model.

[6]The existence of the limit is guaranteed because of t's continuity in d (implied by f's continuity) and Berge's theorem of maximum. The limit in the definition of s^f below exists for the same reason.

with the subscript $y \sim f$ expressing that the probability is computed assuming that the point y is drawn according to the distribution f. Note that $a^f \geq 1$ always holds and $a^f = 1$ if f is the uniform distribution. The number $t^f(w, d, S)$ is the probability that y is within the distance of d from w with respect to all the dimensions in S conditional on the event that these two points are neighbors to each other. Thus a^f is the maximum with respect to w of the ratio of the maximum of such $t^f(w, d, S)$ with respect to S to the minimum.

The measure a^f itself is difficult to handle, but it has an upper bound that is easier to deal with,

$$a^f \leq \frac{1 - ({}_mC_k)s^f}{s^f} := A^f,$$

where

$$s^f = \lim_{\hat{d} \to 0} \min_{w \in [0,1]^m} \min_{S \subseteq \{1,\ldots,m\}, |S|=k} t^f(w, d, S).$$

Here, s^f is the minimum share of neighbors that some collection of dimensions of size k has. The formula follows because s^f is the minimum, so given this value, the most extreme case is that all but one collection of k dimensions have exactly this minimum share, and that only that one dimension has the biggest share.[7]

Now we consider comparative statics with respect to m and k. Denote the limit clustering coefficient as $\hat{d} \to 0$ given m, k, and a distribution f by $\mathrm{Cl}^*(m, k, f)$. In the proposition below, we denote $\binom{m}{k}$ by ${}_mC_k$.

Proposition A1. *(i) Suppose that $k < m' < m$. Let $g \in \Delta([0,1]^{m'})$ and $f \in \Delta([0,1]^m)$. Then $\mathrm{Cl}^*(m', k, g) \geq \mathrm{Cl}^*(m, k, f)$ if*

$$A^f \leq 1 + \frac{{}_mC_k \sqrt{\dfrac{{}_mC_k - {}_{m'}C_k}{{}_{m'}C_k({}_mC_k - 1)}}}{1 - \sqrt{\dfrac{{}_mC_k - {}_{m'}C_k}{{}_{m'}C_k({}_mC_k - 1)}}}.$$

(ii) Suppose that under uniform distribution, $\mathrm{Cl}^*(m, k) \geq \mathrm{Cl}^*(m, k')$. Let $g \in \Delta([0,1]^{m'})$ and $f \in \Delta([0,1]^m)$. Then $\mathrm{Cl}^*(m, k, g) \geq \mathrm{Cl}^*(m, k', f)$ if $k' = m$ or

$$A^f \leq 1 + \frac{{}_mC_{k'} \sqrt{\dfrac{{}_mC_{k'}\left(\dfrac{3}{4}\right)^{k-k'} - {}_mC_k}{({}_mC_{k'} - 1){}_mC_k}}}{1 - \sqrt{\dfrac{{}_mC_{k'}\left(\dfrac{3}{4}\right)^{k-k'} - {}_mC_k}{({}_mC_{k'} - 1){}_mC_k}}}.$$

[7] If f is uniform, $s^f = \frac{1}{{}_mC_k}$ and so $r(m, m', k, s^f)$ reduces to $\frac{1}{{}_{m'}C_k \cdot [1/{}_mC_k]} = {}_mC_k / {}_{m'}C_k$, as expected.

	m		
$k=1$	2	3	4
$m'=1$	4	6.464	8.899
$m'=2$	–	3	4.450
$m'=3$	–	–	2.667

	m	
$k=2$	3	4
$m'=2$	5.346	10.658
$m'=3$	–	4.250

	m
$k=3$	4
$m'=3$	7.899

TABLE 1. Robustness of comparative statics with respect to m. Each entry is the A^f value for the given parameter values k, m', and m that appear in part (i) of Proposition A1. Note that $\mathrm{Cl}^*(m,k)$ is decreasing in m.

	k
$m=2$	2
$k'=1$	5.828

	k	
$m=3$	1	3
$k'=2$	3.070	12.325
$k'=1$	–	5.252

	k		
$m=4$	3	1	4
$k'=2$	2.127	5.854	14.305
$k'=3$	–	5.150	26.856
$k'=1$	–	–	4.055

TABLE 2. Robustness of comparative statics with respect to k. Each entry is the A^f value for the given parameter values m, k', and k that appear in part (ii) of Proposition A1. Note that $\mathrm{Cl}^*(2,1) < \mathrm{Cl}^*(2,2)$, $\mathrm{Cl}^*(3,2) < \mathrm{Cl}^*(3,1) < \mathrm{Cl}^*(3,3)$, and $\mathrm{Cl}^*(4,2) < \mathrm{Cl}^*(4,3) < \mathrm{Cl}^*(4,1) < \mathrm{Cl}^*(4,4)$.

That is, as long as the asymmetry measure A^f is less than the threshold that depends on k, m, and m', the comparative-statics results in Corollary 1 are valid. The proof is based on a straightforward modification of the proof for the main result and, hence, is omitted. In Tables 1 and 2, we provide values of A^f for various (m,m') pairs and fixed ks. As can be seen from the tables, the comparative statics go through as long as the distribution is not too asymmetric.

Now we apply this result to compare the limit clustering coefficients under the Min and Max norms. First, the comparative statics from the case with the uniform distribution always carry over if $k'=m$.[8] Thus, we concentrate on the case with $m<9$, in which $k=m$ and $k'=1$ hold. Hence, the inequality in part (ii) of Proposition A1 becomes

$$A^f \le \frac{1+\sqrt{m\left(\frac{3}{4}\right)^{m-1}-1}}{1-\sqrt{\dfrac{m\left(\frac{3}{4}\right)^{m-1}-1}{m-1}}}.$$

[8]More generally, given some (k,m) pair, if it is true under the uniform distribution that the clustering is above that given by the Max norm, then it is true for any other distribution. The reason is that under the Max norm, for any small $\epsilon>0$, the set of the neighbors of any point $x\in X$ is a subset of the ϵ-neighborhood of x with respect to the Euclidean distance if $\hat{d}>0$ is small enough. Since any distribution that has a density function is locally uniform, changes in the distribution cannot affect the comparative-statics result.

	$k = m$						
	2	3	4	5	6	7	8
$k' = 1$	3.414	3.560	3.033	2.487	2.037	1.663	1.294

TABLE 3. Robustness of comparative statics of the comparison of Min and Max norms. Each entry is the A^f value for the given parameter values m, $k' = 1$ and $k = m$ that appear in part (ii) of Proposition A1. Note that when $m < 9$, $\mathrm{Cl}^*(m, 1) < \mathrm{Cl}^*(m, m)$.

Table 3 summarizes the value of A^f for various ms. It suggests that a broad range of distributions is consistent with the comparison of the Min and Max norms under the uniform distribution, especially when $m < 9$ is small.

E.3 *Clustering lower bound and triangle inequality*

Define social distance by $d(x_i, x_j) = \|x_i - x_j\|$, where $\|\cdot\|$ satisfies the standard norm axioms in $\mathbb{R}^m$. That is, for all $\alpha \in \mathbb{R}$ and $y, y' \in \mathbb{R}^m$, it satisfies (i) absolute homogeneity, i.e., $|\alpha|\|y\| = \|\alpha y\|$, (ii) triangle inequality, i.e., $\|y + y'\| \leq \|y\| + \|y'\|$, and (iii) separates points, i.e., if $\|y\| = 0$, then y is the zero vector. Below we show that under such social distance d, $C_m := \int_0^1 mq^{m-1}(1-q)^m \, dq > 0$ is a lower bound of the limit clustering coefficient.

To see this, fix type x_i of an agent i such that its $\hat{d}$-neighborhood measured by social distance $d(\cdot, \cdot)$ is contained in the interior of X. Let x_j and x_k denote two randomly chosen points in the neighborhood of x_i by the uniform distribution. By the triangle inequality, agents j and k with type x_j and x_k, respectively, are connected if

$$d(x_i, x_j) + d(x_i, x_k) \leq \hat{d}. \tag{A2}$$

We claim that the standard-norm axioms imply a strictly positive lower bound C_m of the probability that this inequality (A2) is satisfied. To see this, first we note that, for any $r \in [0, 1]$, the Lebesgue measure of the $(r\hat{d})$-neighborhood of x_i is r^m of the $\hat{d}$-neighborhood.[9] This implies that, for each $r \in [0, 1]$, $d(x_i, x_j) \leq r\hat{d}$ holds with probability r^m for small enough $\hat{d} > 0$. Hence, the probability density of the variable $q := \frac{d(x_i, x_j)}{\hat{d}}$, conditional on x_j being in the $\hat{d}$-neighborhood of x_i, is mq^{m-1}. Therefore, (A2) is satisfied with probability $\int_0^1 mq^{m-1}(1-q)^m \, dq = C_m$. Hence, the limit clustering coefficient

[9]To see this, observe that $d(x, (1-r)x + rx') = rd(x, x')$ for any points $x, x' \in X$ by the absolute homogeneity axiom. This ensures that a boundary point of the $(r\hat{d})$-neighborhood of x can be written as $(1-r)x + rx'$ for some boundary point x' of the $\hat{d}$-neighborhood. Geometrically, the $(r\hat{d})$-neighborhood is equal to the $\hat{d}$-neighborhood proportionally scaled by the factor of r. This implies that if x belongs to the interior of X, then its $\hat{d}$-neighborhood is contained in the interior of X for all sufficiently small $\hat{d} > 0$. This is because otherwise there exists a point $x' \neq x$ that belongs to the $\hat{d}$-neighborhood of x for any $\hat{d} > 0$, which implies $\|x - x'\| = 0$, a contradiction to separates points.

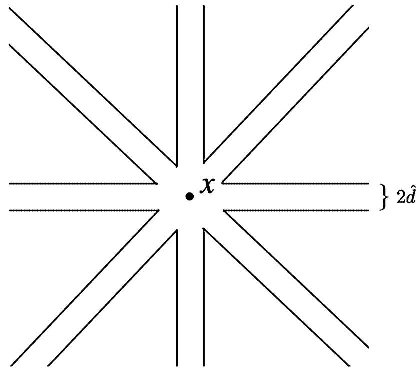

FIGURE A3. The $\hat{d}$-neighborhood of x when $m = 2$ and $T = 2$ in Example 1.

is bounded by $C_m > 0$ from below.[10] However, such a lower bound cannot be established if we relax the standard-norm axioms, and the clustering coefficient can be arbitrarily close to 0 in general. The example below illustrates this point by using a generalized version of the Min norm.

EXAMPLE 4. Let $m \geq 2$. Fix a finite collection of $m \times m$ regular matrices $\{A(1), A(2), \dots, A(T)\}$. For each $t = 1, \dots, T$, the set of vectors $\{e_l A(t)\}_{l=1,\dots,m}$ spans $\mathbb{R}^m$, where each $e_l = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the lth unit vector in $\mathbb{R}^m$. Therefore, for any vector $y \in \mathbb{R}^m$ and $t = 1, \dots, T$, there is a unique collection of coefficients $\{\alpha_l^y(t)\}_{l=1,\dots,m}$ that satisfies $y = \sum_{l=1,\dots,m} \alpha_l^y(t) e_l A(t)$.

Define social distance by

$$d(x, x') = \min_{t=1,\dots,T} \min_{l=1,\dots,m} \left|\alpha_l^{x-x'}(t)\right|.$$

For any number $w > 0$, the assumption that $m \geq 2$ implies that we can choose T large and appropriate matrices $\{A(t)\}_{t=1,\dots,T}$ such that, for any $t, t' = 1, \dots, T$ and $l, l' = 1, \dots, m$, $e_l A(t)$ and $e_{l'} A(t')$ are not proportional to each other, and the limit clustering coefficient is below w. Intuitively, i's neighborhood consists of $m \times T$ "stripes" and the types of two neighbors j and k of agent i typically belong to different stripes in a $\hat{d}$-neighborhood of i's type when $m \times T$ is large, and therefore, j and k are unlikely to be connected. Figure A3 illustrates an example of the $\hat{d}$-neighborhood when $m = 2$ and $T = 2$, which consists of four stripes. This norm satisfies absolute homogeneity but violates triangle inequality and separates points. Note that the Min norm is a special case of $T = 1$ where $A(t)$ is the identity matrix. $\diamondsuit$

[10] A better bound is given by $(1 - (1/2)^m)C_m + (1/2)^m$. This is because with probability $(\frac{1}{2})^m$, $\text{sgn}(x_{il} - x_{jl}) = \text{sgn}(x_{il} - x_{kl})$ holds for every dimension $l = 1, \dots, m$. In such a case, j and k are connected independently of the distances $d(x_i, x_j)$ and $d(x_i, x_k)$.

E.4 *Discrete type space model*

In the main part of this paper we assumed that agents are distributed over the type space $[0,1]^m$, and that the distribution of agents on this space is given by some strictly positive density function. In practice, it may be better to regard some dimensions, such as gender, as taking discrete values. In this subsection, we consider a variant of the main model in which the type space is discrete, to examine whether our main results are robust to such modification of the model. We first discuss the subtleness in constructing a model with discrete type space, and then show that our main qualitative results go through in an appropriately defined model with discrete type space.

As we mentioned in Section 6, the limit analysis as $\hat{d} \to 0$ enables us to implement meaningful comparative statics under the main model, as we can take that limit with keeping the expected degrees identical across two different networks with different k's and m's. This argument does not go through when we have a discrete type space because in a discrete model, when the cutoff is near zero, an agent's only neighbors are those with distance zero, and thus, without any additional assumptions, we cannot set the expected degrees to be identical across two different networks with different k's and m's.

To highlight the effect of discreteness, we consider the simplest form of discrete type space, $X = \{0,1\}^m$, and a unit mass of agents distributed uniformly over X, so that at each point in this space, measure $\frac{1}{2^m}$ of agents exists. Given m and k, agent i at point $x \in X$ is a *potential neighbor* of agent j at $y \in X$ if x and y have at least k common attributes. For example, if $m = 5$, $x = (0,0,1,0,1)$, and $y = (1,0,1,1,0)$, they are potential neighbors with each other if $k \le 2$, but otherwise not. Let the size of potential neighbors for each agent (which we assume is identical for all agents) be M. To overcome the difficulty described in the previous paragraph, we assume that an agent can be linked with only a subset of potential neighbors where the size of this subset may differ for different (k,m) values. Specifically, we parameterize the model by *size of neighbors*, denoted by $p > 0$. That is, an agent is linked to a potential neighbor independently with probability $\frac{p}{M}$ and is not linked to anyone outside the set of potential neighbors, so that the size of neighbors is p. For simplicity, we assume that p is sufficiently small so that the probability $\frac{p}{M}$ is well defined (i.e., no more than 1).

Let g be the network generated by the rule described above, and let $\mathrm{Cl}(m,k,p)$ and $\mathrm{APL}(m,k,p)$ be the clustering coefficient and the average path length of g, respectively, given m, k, and p. Denote the binomial coefficient $\binom{a}{b}$ by $C(a,b)$. The next proposition states the formulas for $\mathrm{Cl}(m,k,p)$ and $\mathrm{APL}(m,k,p)$.

PROPOSITION A2. *(i) The following equality holds:*

$$\mathrm{Cl}(m,k,p) = \left(\sum_{0 \le s,t \le m-k} \frac{P_{st} C(m,s) C(m,t)}{\left(\sum_{0 \le i \le m-k} C(m,i) \right)^2} \right) \cdot \frac{2^m}{\sum_{k \le l \le m} C(m,l)} p,$$

where

$$P_{st} = P_{ts} = \begin{cases} 1 & \text{if } s+t \le m-k, \\ \dfrac{\sum_{0 \le h \le \frac{m-k-s+t}{2}} C(s,t-h)C(m-s,h)}{C(m,t)} & \text{if } s+t > m-k \text{ and } s \ge t. \end{cases}$$

(ii) The following equality holds:

$$\mathrm{APL}(m,k,p) = -p + \sum_{l=1}^{m} \frac{C(m,l)}{2^m} \max\big[2, D(l,m,k)\big],$$

where $D(l,m,k) = \min\{a \in \mathbb{N} | a \ge \frac{l}{m-k}\}$.

The clustering coefficient is the product of two terms: The first is the term in the parentheses, which is the probability that an agent's two neighbors are potential neighbors with each other. The second is the probability that they are actually connected ($\frac{p}{M}$). Notice that for fixed m, as k becomes larger, the size of potential neighbors (M) decreases, so $\frac{p}{M}$ is increasing in k.

Proof of Proposition A2. Part (i). Take arbitrarily agents i, j, and k situated at points x, y, and z in the type space X, respectively. First we compute the conditional probability of j and k being potential neighbors of each other, given $ij \in g$ and $ik \in g$. This probability depends on the type difference between i and j, as well as i and k. Let s (resp. t) denote the number of different entries between x and y (resp. x and z). Note that $0 \le s, t \le m-k$, because $ij \in g$ and $ik \in g$. Due to the uniform distribution assumption, given $ij \in g$, conditional distribution of s follows probability mass function $h(s) := \frac{C(m,s)}{\sum_{i=0}^{m-k} C(m,i)}$. Conditional distribution of t follows the same probability mass function.

Let P_{st} denote the conditional probability of j and k being potential neighbors of each other, given s and t. Since we have $P_{st} = P_{ts}$, it is sufficient to focus on the case $s \ge t$. If $s + t \le m - k$, then it is clear to see that the type difference between j and k is always within $m-k$ entries, so that $P_{st} = 1$.

Consider the case where $s+t > m-k$. It is without loss of generality to suppose $x = (0,\dots,0)$ and $y = (1,\dots,1,0,\dots,0)$, where the first s entries are all 1 and the others are 0. For each $x', x'' \in X$, define $\|x' - x''\| := \sum_{i=1}^{m} |x'_i - x''_i|$. Since the conditional distribution of z is uniform in $\{x' \in X : \|x'\| = t\}$, we have

$$P_{st} = \frac{\sum_{r=0}^{m-k} \sharp\big\{z \in X : \|z\| = t, \|z-y\| = r\big\}}{C(m,t)}.$$

We have that

$$
\begin{aligned}
&\sharp\{z \in X : \|z\| = t, \|z - y\| = r\} \\
&\quad = \begin{cases} 0 & \text{if } r < s - t \text{ or } r = s - t + 2l + 1 \text{ for some } l = 0, 1, 2, \ldots, \\ C(s, t - h)C(m - s, h) & \text{if } r = s - t + 2l \text{ for some } l = 0, 1, 2, \ldots. \end{cases}
\end{aligned}
$$

To see this, first consider the case $z = z' := (1, \ldots, 1, 0, \ldots, 0)$, where the first t entries are all 1 and the others are 0. Clearly this point belongs to $\arg\min_{\|z\|=t}\{\|z - y\|\}$. Since $\|z' - y\| = s - t$ holds, $\|z - y\| < s - t$ is not possible, given $\|z\| = t$. Next, starting from k's type being z', modify his type arbitrarily, keeping at t the difference between 0 and this type. Notice that $\|z - y\| - (s - t)$ cannot be an odd number, given $\|z\| = t$. If there exists a nonnegative integer l such that we can write $\|z - y\| = s - t + 2l$, we can find $C(s, t - l)C(m - s, l)$ patterns of k's type z such that $\|z\| = t$.

Then, by aggregating P_{st} over every (s, t) pair with the probability mass function $h(\cdot)$, we obtain the conditional probability of j and k being potential neighbors of each other, given $ij \in g$ and $ik \in g$.

Next we compute the probability of a link being formed between two agents who are potential neighbors of each other. This probability $p(k, m)$ has to be adjusted to keep the size of neighbors p independent of k and m. Given k and m, the size of potential neighbors is $\frac{\sum_{k \le l \le m} C(m,l)}{2^m}$. Thus, we obtain $p(k, m) = \frac{2^m}{\sum_{k \le l \le m} C(m,l)} p$.

Finally, the clustering coefficient is obtained by multiplying the $p(k, m)$ and the conditional probability of j and k being potential neighbors of each other, given $ij \in g$ and $ik \in g$, which yields the desired formula.

Part (ii). First note that if two agents are linked by path length 2 or more in the model with $p = 1$ (i.e., they are not potential neighbors with each other), then they are linked by the same path length in any model with $p > 0$. To see this, note that there is positive mass of population of each type; thus, for any given sequence of types $(x_1, x_2, \ldots, x_n)$, there are some sequence of agents $(i_1, i_2, \ldots, i_n)$ such that each agent i_l is of type x_l, and i_l and i_{l+1} are linked with each other for all $l = 1, \ldots, n-1$. Thus the average path length is

$$
\mathrm{APL}(m, k, p) = M \cdot \left[\frac{p}{M} \cdot 1 + \left(1 - \frac{p}{M}\right) \cdot 2\right] + \sum_{l=m-k+1}^{m} \frac{C(m, l)}{2^m} D(l, m, k),
$$

where $D(l, m, k) = \min\{a \in \mathbb{N} | a \ge \frac{l}{m-k}\}$. Here the first term is the size of potential neighbors of an agent times the average path length from the agent in consideration to these agents.

This is equal to

$$
-p + 2M + \sum_{l=m-k+1}^{m} \frac{C(m, l)}{2^m} D(l, m, k)
$$

or

$$
-p + \sum_{l=0}^{m} \frac{C(m, l)}{2^m} \max\bigl[2, D(l, m, k)\bigr].
$$

$m \setminus k$	1	2	3	4	5
1	p	–	–	–	–
2	$1.037p$	$4p$	–	–	–
3	$0.863p$	$1.625p$	$8p$	–	–
4	$0.811p$	$0.805p$	$2.688p$	$16p$	–
5	$0.796p$	$0.539p$	$1.297p$	$2.370p$	$32p$

TABLE 4. $\mathrm{Cl}(m, k, p)$.

$m \setminus k$	1	2	3	4	5
1	∞	–	–	–	–
2	$2-p$	∞	–	–	–
3	$2-p$	$2.125-p$	∞	–	–
4	$2-p$	$2-p$	$2.375-p$	∞	–
5	$2-p$	$2-p$	$2.031-p$	$2.656-p$	∞

TABLE 5. $\mathrm{APL}(m, k, p)$.

This completes the proof. □

In Tables 4 and 5 we provide examples of values $\mathrm{Cl}(m, k, p)$ and $\mathrm{APL}(m, k, p)$ where $m \in \{1, 2, 3, 4, 5\}$.

The results in the tables feature the key properties we obtained in Corollaries 1 and 2. That is, (i) the clustering coefficient under the Max norm is higher than under the Min norm, (ii) the clustering coefficient is not monotonic in k while it is decreasing in m, and (iii) the average path length is increasing in k while decreasing in m.

E.5 *Approximation results*

In this section, we consider the cutoff rule model where the number of agents n is finite and the cutoff $\hat{d}$ is positive. The following result gives bounds of the deviations of the expected values of the clustering coefficient and the average path length from the limit values obtained in the main section.

PROPOSITION A3. *(i) For $n \geq 3$, $|\mathrm{Cl}^* - E[\mathrm{Cl}]| = O(\hat{d})$ as $\hat{d} \to 0$.*

(ii) For $k < m$, $-O(\hat{d}) \leq E[\mathrm{APL}] - \mathrm{APL}^* \leq O(e^{-n\hat{d}^m} n)$ as $(n, \hat{d}) \to (\infty, 0)$.

This result helps us to interpret the limit values Cl^* and APL^* in the main section. That is, it bounds the orders of n and d that we need so as to approximate the limit values at a given precision. We note that the bound of clustering does not depend on n. This is because the clustering coefficient is an expectation of independent probabilities.

Proof of Proposition A3. Part (i). Fix $n \geq 3$. We consider the sequence of models with varying $\hat{d} \to 0$. First we have

$$\begin{aligned}
E[\mathrm{Cl}] &= E[\mathrm{Cl}_i] \\
&= \left(1-(1-2\hat{d})^m\right)E\left[\mathrm{Cl}_i : x_i \notin X(\hat{d})\right] + (1-2\hat{d})^m E\left[\mathrm{Cl}_i : x_i \in X(\hat{d})\right] \\
&= O(\hat{d}) + O(1-\hat{d})E\left[\mathrm{Cl}_i : x_i \in X(\hat{d})\right],
\end{aligned}$$

where $X(\hat{d})$ is the set of types that are bounded away (for each dimension) from the boundary by $\hat{d}$. Let $S(i,j) = \{l : |x_{il} - x_{jl}| \leq \hat{d}\} \subseteq \{1,2,\dots,m\}$ be the set of dimensions on which i and j are close within the distance of $\hat{d}$. We have

$$\begin{aligned}
&E\left[\mathrm{Cl}_i : x_i \in X(\hat{d})\right] \\
&= \Pr\left[jh \in g : x_i \in X(\hat{d}), \{ij, ih\} \subseteq g\right] \\
&=: \hat{\Pr}[jh \in g] \\
&= \sum_{l=k}^{m} \hat{\Pr}\left[\left|S(i,j)\right| = l\right]\hat{\Pr}\left[jh \in g : \left|S(i,j)\right| = l\right] \\
&= \frac{1}{\mu\left(B_{\hat{d}}(x_i) \cap X(\hat{d})\right)} \sum_{l=k}^{m} \frac{m!}{(m-l)!l!}(2\hat{d})^l(1-4\hat{d})^{m-l}\hat{\Pr}\left[jh \in g : \left|S(i,j)\right| = l\right] \\
&= O(1-\hat{d})\hat{\Pr}\left[jh \in g : \left|S(i,j)\right| = k\right] + O(\hat{d}),
\end{aligned}$$

where $\hat{\Pr}[\cdot]$ denotes the probability measure conditional on $x_i \in X(\hat{d})$ and $\{ij, ih\} \subseteq g$.

Then

$$\begin{aligned}
&\hat{\Pr}\left[jh \in g : \left|S(i,j)\right| = k\right] \\
&= \sum_{l=k}^{m} \hat{\Pr}\left[\left|S(i,h)\right| = l : \left|S(i,j)\right| = k\right]\hat{\Pr}\left[jh \in g : \left|S(i,j)\right| = k, \left|S(i,h)\right| = l\right] \\
&= \frac{1}{\mu\left(B_{\hat{d}}(x_i) \cap X(\hat{d})\right)} \\
&\quad \times \sum_{l=k}^{m} \frac{m!}{(m-l)!l!}(2\hat{d})^l(1-4\hat{d})^{m-l}\hat{\Pr}\left[jh \in g : \left|S(i,j)\right| = k, \left|S(i,h)\right| = l\right] \\
&= O(1-\hat{d})\hat{\Pr}\left[jh \in g : \left|S(i,j)\right| = \left|S(i,h)\right| = k\right] + O(\hat{d}).
\end{aligned}$$

Next

$$
\begin{aligned}
&\hat{\Pr}\big[jh \in g : |S(i,j)| = |S(i,h)| = k\big] \\
&\quad = \hat{\Pr}\big[S(i,j) = S(i,h) : |S(i,j)| = |S(i,h)| = k\big] \\
&\qquad \times \hat{\Pr}\big[jh \in g : S(i,j) = S(i,h), |S(i,j)| = |S(i,h)| = k\big] \\
&\qquad + \hat{\Pr}\big[S(i,j) \neq S(i,h) : |S(i,j)| = |S(i,h)| = k\big] \\
&\qquad \times \hat{\Pr}\big[jh \in g : S(i,j) \neq S(i,h), |S(i,j)| = |S(i,h)| = k\big] \\
&\quad = \frac{(m-k)!k!}{m!} \hat{\Pr}\big[jh \in g : S(i,j) = S(i,h), |S(i,j)| = |S(i,h)| = k\big] \\
&\qquad + \left(1 - \frac{(m-k)!k!}{m!}\right) \hat{\Pr}\big[jh \in g : S(i,j) \neq S(i,h), |S(i,j)| = |S(i,h)| = k\big].
\end{aligned}
$$

Regarding the last line, we can get bounds:

$$
\begin{aligned}
\left(\frac{3}{4}\right)^k &\leq \hat{\Pr}\big[jh \in g : S(i,j) = S(i,h), |S(i,j)| = |S(i,h)| = k\big] \\
&\leq \left(\frac{3}{4}\right)^k + \left(1 - \left(\frac{3}{4}\right)^k\right)\left(1 - \left(\frac{1-4\hat{d}}{1-2\hat{d}}\right)^{m-k}\right) \\
&= \left(\frac{3}{4}\right)^k + O(\hat{d}),
\end{aligned}
$$

where $(\frac{3}{4})^k$ represents the conditional probability that j and h are close to each other in dimensions $S(i,j)$. The second term on the second line gives a probability bound of the other possibilities that j and h are linked. Also we have

$$
\begin{aligned}
\hat{\Pr}\big[jh \in g : S(i,j) \neq S(i,h), |S(i,j)| = |S(i,h)| = k\big] &\leq \left(1 - \left(\frac{1-4\hat{d}}{1-2\hat{d}}\right)^{m-k}\right) \\
&= O(\hat{d}).
\end{aligned}
$$

Therefore, what we have shown is the inequality of the form

$$
\alpha \mathrm{Cl}^* \leq E[\mathrm{Cl}] \leq \alpha\big(\mathrm{Cl}^* + O(\hat{d})\big),
$$

where coefficient α is such that $1 - \alpha = O(\hat{d})$. This shows that $|\mathrm{Cl}^* - E[\mathrm{Cl}_i]| = O(\hat{d})$.

Part (ii). With fixed $k < m$, we look at a sequence of models with varying $(n, \hat{d}) \to (\infty, 0)$.

First we have

$$
\begin{aligned}
&E[\mathrm{APL}] \\
&\quad = E[\mathrm{PL}_{ij} : \mathrm{PL}_{ij} < \infty] \\
&\quad = \Pr\big[\forall l |x_{il} - x_{jl}| > \hat{d} : \mathrm{PL}_{ij} < \infty\big] E\big[\mathrm{PL}_{ij} : \forall l |x_{il} - x_{jl}| > \hat{d}, \mathrm{PL}_{ij} < \infty\big] \\
&\qquad + \Pr\big[\exists l |x_{il} - x_{jl}| \leq \hat{d} : \mathrm{PL}_{ij} < \infty\big] E\big[\mathrm{PL}_{ij} : \exists l |x_{il} - x_{jl}| \leq \hat{d}, \mathrm{PL}_{ij} < \infty\big]
\end{aligned}
$$

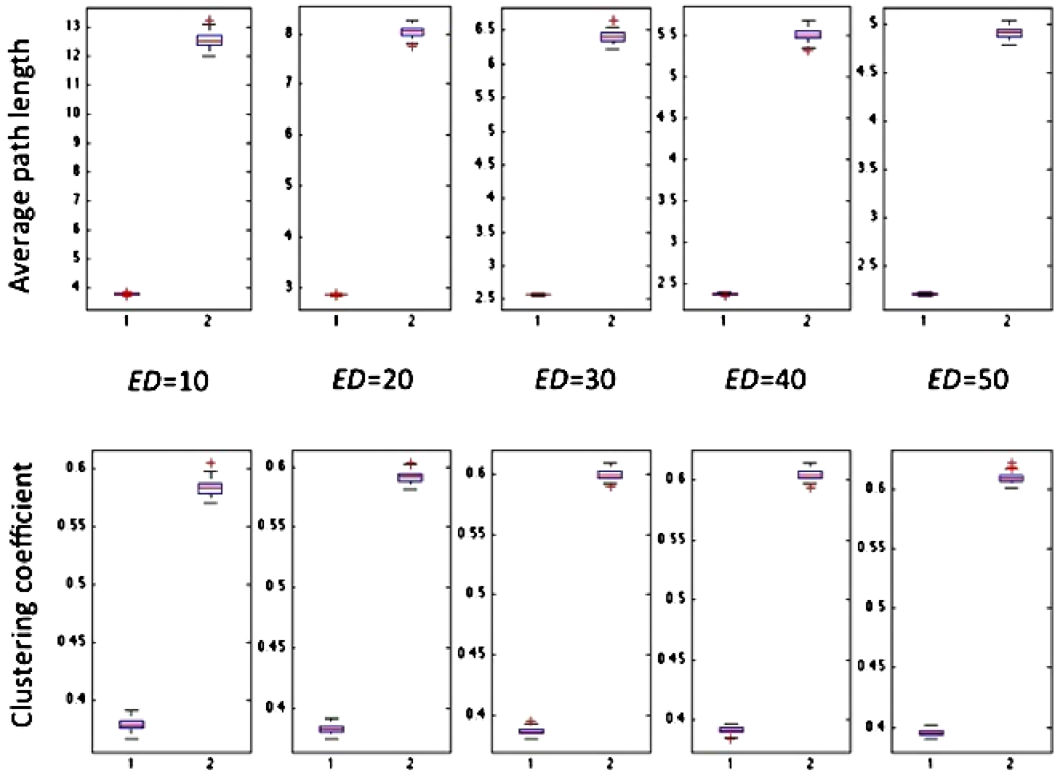

FIGURE A4. Comparative statics under $m = 2$.

and

$$\Pr\big[\forall l |x_{il} - x_{jl}| > \hat{d} : \mathrm{PL}_{ij} < \infty\big] > (1 - 2\hat{d})^m = 1 - O(\hat{d}),$$

$$E\big[\mathrm{PL}_{ij} : \exists l |x_{il} - x_{jl}| \le \hat{d}, \mathrm{PL}_{ij} < \infty\big] \le E\big[\mathrm{PL}_{ij} : \forall l |x_{il} - x_{jl}| > \hat{d}, \mathrm{PL}_{ij} < \infty\big].$$

Below we use $\tilde{E}$ and $\tilde{\Pr}$ to denote the expectation and probability conditional on $|x_{il} - x_{jl}| > \hat{d}$ for every l and $\mathrm{PL}_{ij} < \infty$. First note that $\tilde{\Pr}[\mathrm{PL}_{ij} < \mathrm{APL}^*] = 0$ because it is impossible to find an indirect path that connects these two agents with steps less than APL^* when $|x_{il} - x_{jl}| > \hat{d}$ for every l. Therefore, $\tilde{E}[\mathrm{PL}_{ij}] = \sum_{l=\mathrm{APL}^*}^{n} \tilde{\Pr}[\mathrm{PL}_{ij} = l]l$. Here $\tilde{\Pr}[\mathrm{PL}_{ij} = \mathrm{APL}^*]$ is more than

$$\alpha' := \big(1 - \big(1 - (\hat{d})^m\big)^{n-2}\big)^{\mathrm{APL}^*} = 1 - O\big[\big(1 - \hat{d}^m\big)^n\big],$$

which is a probability bound that an indirect path with length APL^* (i.e., β in the proof of Theorem 2) exists. Note that this construction relies on the fact that $k < m$.

Hence, we have shown the inequality of the form

$$\begin{aligned} &O(\hat{d})1 + \big(1 - O(\hat{d})\big)\mathrm{APL}^* \\ &\quad \le E[\mathrm{APL}] \\ &\quad \le \big(O(\hat{d}) + \big(1 - O(\hat{d})\big)\big)\big(\alpha'\mathrm{APL}^* + (1 - \alpha')n\big). \end{aligned}$$

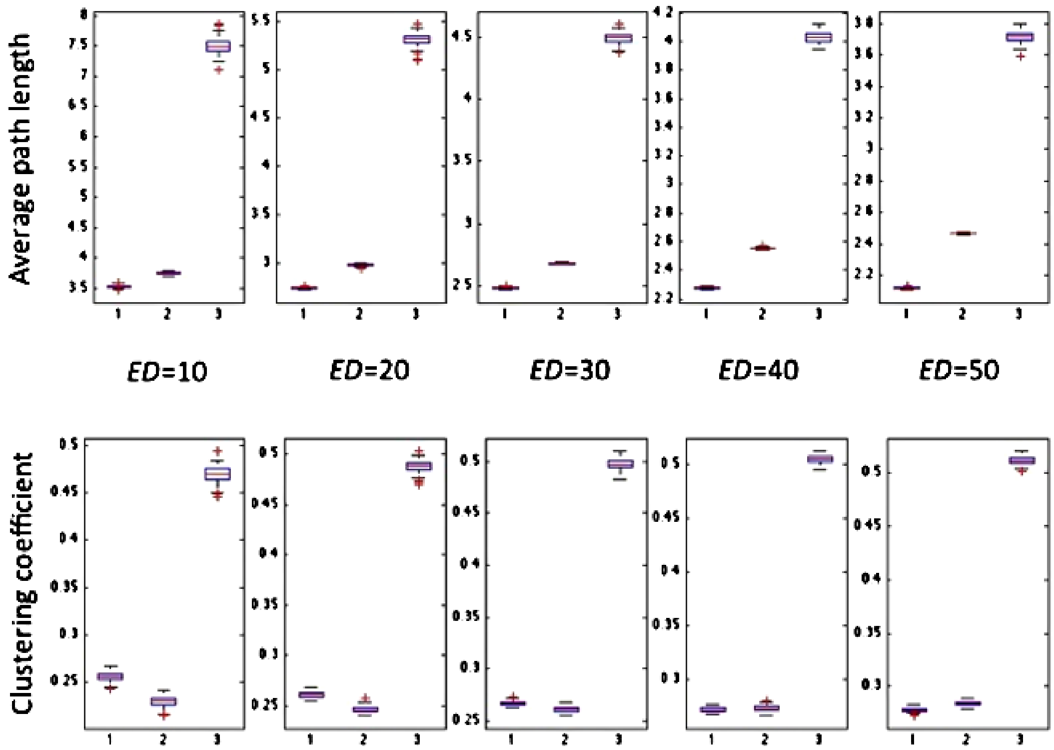

FIGURE A5. Comparative statics under $m = 3$.

This inequality, together with the fact that $(1 - d^m)^{1/d^m} \to e^{-1}$, results in the desired formula. $\square$

The extent to which our results regarding the limit values are economically meaningful depends on the robustness of the comparative-statics results. Here we consider the comparative statics of clustering coefficient and average path length when n is finite and $\hat{d}$ is positive. Figures A4 and A5 plot the realizations of clustering coefficients and average path lengths of 100 generations of networks for each parameter combination where $m \in \{2, 3\}$ and $n = 1000$ under the uniform type distribution. For each profile (m, k), we adjusted the cutoff level to generate five different levels of expected degrees,[11] $\text{ED} \in \{10, 20, 30, 40, 50\}$ and ran simulations. The x-axis of each figure corresponds to the values of k. Each diagram shows the 0.25, 0.50, and 0.75 fractiles of the resulting distribution, along with the outliers.

Our results in the main sections show that $\text{Cl}^*(2,2) > \text{Cl}^*(1,2)$ and $\text{APL}^*(2,2) > \text{APL}^*(1,2)$ when $m = 2$, and $\text{Cl}^*(3,3) > \text{Cl}^*(1,3) > \text{Cl}^*(2,3)$ and $\text{APL}^*(3,3) > \text{APL}^*(2,3) > \text{APL}^*(1,3)$ when $m = 3$. Under broad parameter combinations, the simulation results we ran are consistent with our comparative statics of the limit values. In Figure A2, the simulation results on the clustering coefficients under $m = 3$ are inconsistent with the comparative statics of the limit values when the expected degree is

[11] To be more precise, this is the expected degree of an agent whose type belongs to $[\hat{d}, 1 - \hat{d}]^m \subset X$.

relatively high. This is because the cutoff value is too high in such cases, so that there can be significant deviations from the limit values of the clustering coefficients.[12]

References

Carlsson, Hans and Eric Van Damme (1993), "Global games and equilibrium selection." *Econometrica*, 61, 989–1018. [3]

Kajii, Atsushi and Stephen Morris (1997), "The robustness of equilibria to incomplete information." *Econometrica*, 65, 1283–1309. [3]

Kandori, Michihiro, George J. Mailath, and Rafael Rob (1993), "Learning, mutation, and long run equilibria in games." *Econometrica*, 61, 29–56. [3]

Young, H. Peyton (1993), "The evolution of conventions." *Econometrica*, 61, 57–84. [3]

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[12]When the expected degree is 40, the cutoff for $k = 2$ is as high as 0.0602. This is so high that two neighbors of an agent can be linked with each other with a nontrivial probability even when they are connected with the agent through different sets of dimensions. This makes the simulated value of the clustering coefficient for $k = 2$ significantly higher than the limit value, leading to the inconsistent comparative statics.